Chapter 8

Symmetric Instability and Rossby Waves

We are now in a position to begin investigating more complex disturbances in a rotating, stratified environment. Most of the disturbances of interest are manifested in low Rossby number flows, and are thus amenable to analysis using the quasi-geostrophic equations. However, one type of disturbance which occurs in shear under these conditions is not represented in these equations. Under the right circumstances this disturbance is unstable, and is given the name *symmetric instability*. The rest of the disturbances of interest constitute various types of *Rossby waves*.

8.1 Symmetric instability

Symmetric instability occurs on a base state in hydrostatic and geostrophic balance. The base state is sheared, which implies a horizontal temperature gradient. The instability draws energy from this temperature gradient, thus tending to reduce the gradient and simultaneously the shear.

We analyze symmetric instability using the hydrostatic Boussinesq equations in geometric coordinates. Since the disturbance takes the form of circulations in the plane normal to the ambient wind, assumed here to point in the x-direction, we can assume $\partial/\partial x = 0$, resulting in the reduced set of governing equations

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8.1}$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = 0 \tag{8.2}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial \pi}{\partial y} + fu = 0 \tag{8.3}$$

$$\frac{\partial \pi}{\partial z} - b = 0 \tag{8.4}$$

$$\frac{\partial b}{\partial t} + v\frac{\partial b}{\partial y} + w\frac{\partial b}{\partial z} = 0. \tag{8.5}$$

The base state is more complex than we have previously discussed, as the base state kinematic pressure and buoyancy depend on both y and z. We split these variables into base state and perturbation parts, $\pi = \pi_0(y,z)+\pi'$, $b = b_0(y,z)+b'$. The base state wind, however, can be taken as a function of z only. We make the simple assumption that $u = Sz+u'$ where S is a constant vertical shear. The other velocity components v and w are assumed to have zero base state parts. The base state equations which need to be satisfied are hydrostatic and geostrophic balance:

$$\frac{\partial \pi_0}{\partial z} - b_0 = 0 \qquad \frac{\partial \pi_0}{\partial y} + fSz = 0. \tag{8.6}$$

We assume in addition that the Brunt-Väisälä frequency

$$\frac{\partial b_0}{\partial z} = N^2 \tag{8.7}$$

is constant. Cross-differentiating the two balance conditions in equation (8.6) to eliminate π_0 results in the thermal wind condition on b_0:

$$\frac{\partial b_0}{\partial y} = -fS. \tag{8.8}$$

A solution to this set of equations is

$$\pi_0 = -fSyz + N^2z^2/2 \qquad b_0 = -fSy + N^2 z, \tag{8.9}$$

as is easily verified by direct substitution. Both kinematic pressure and buoyancy decrease in the $+y$ direction in response to the sheared wind.

Linearizing the governing equations about this base state results in

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8.10}$$

$$\frac{\partial u'}{\partial t} + Sw - fv = 0 \tag{8.11}$$

$$\frac{\partial v}{\partial t} + \frac{\partial \pi'}{\partial y} + fu' = 0 \tag{8.12}$$

$$\frac{\partial \pi'}{\partial z} - b = 0 \tag{8.13}$$

$$\frac{\partial b'}{\partial t} - fSv + N^2 w = 0. \tag{8.14}$$

The novel aspect of this set of equations is the advection of the y gradient of ambient buoyancy by the y component of the perturbation wind. Assuming a solution of the form $\exp\left[i\left(ly + mz - \omega t\right)\right]$ results in the homogeneous set of equations

$$\begin{pmatrix} 0 & il & im & 0 & 0 \\ -i\omega & -f & S & 0 & 0 \\ f & -i\omega & 0 & il & 0 \\ 0 & 0 & 0 & im & -1 \\ 0 & -fS & N^2 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} u' \\ v \\ w \\ \pi' \\ b' \end{pmatrix} = 0. \tag{8.15}$$

Taking the determinant of the matrix yields the secular equation

$$\omega^2 - \frac{l^2 N^2}{m^2} - f^2 - \frac{2lfS}{m} = 0 \tag{8.16}$$

and adding and subtracting $l^2 S^2/m^2$ from the left side facilitates the completion of a square, resulting in

$$\omega^2 = \frac{l^2 S^2}{m^2}\left(\mathrm{J} - 1\right) + \left(f + \frac{lS}{m}\right)^2 \tag{8.17}$$

where $\mathrm{J} = N^2/S^2$ is the Richardson number.

The square of the frequency ω^2 is guaranteed positive if $\mathrm{J} \geq 1$, which means that the resulting disturbance is neutrally stable. However, if $\mathrm{J} < 1$, the possibility of instability exists, since certain values of the horizontal wavenumber l could result in $\omega^2 < 0$. In particular, a critical value of l,

$$l_C = -mf/S, \tag{8.18}$$

causes the second term on the right side of equation (8.17) to vanish, guaranteeing instability. The Richardson number threshold for instability of $\mathrm{J} = 1$ for symmetric instability is larger than in the necessary condition for instability $\mathrm{J} < 1/4$ for stratified shear flow with no rotation. Thus, symmetric instability manifests itself first as the shear increases.

In the limit of zero shear, equation (8.16) shows that the dispersion relation for these modes reduces to that of hydrostatic inertia-gravity waves:

$$\omega^2 = f^2 + \frac{l^2 N^2}{m^2}. \tag{8.19}$$

When the shear is strong enough to produce instability, the frequency becomes purely imaginary $\omega = i\nu$ and the assumed form of the solution becomes $\exp\left[i(ly + mz)\right]\exp(\nu t)$. Degeneracy exists to the extent that the same frequency arises if l and m are replaced by $-l$ and $-m$. Superimposing the original solution and its degenerate twin with the right phase produces a physically realizable real solution for the vertical velocity:

$$w = W\cos(ly + mz)\exp(\nu t) \tag{8.20}$$

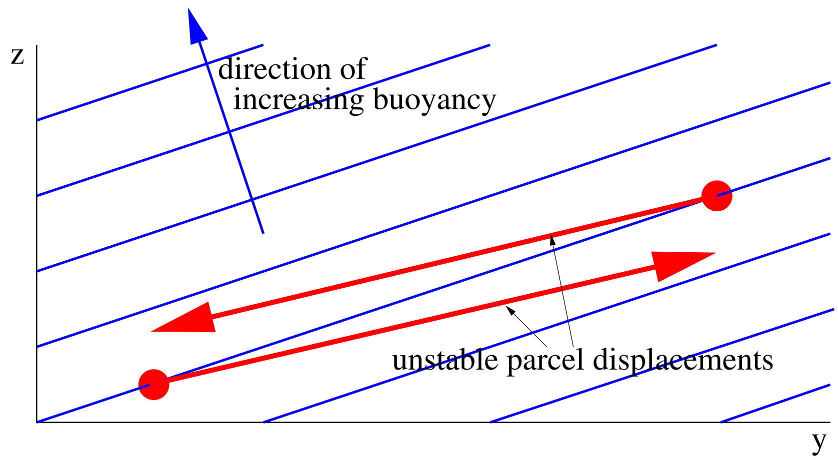

Figure 8.1: Schematic symmetric instability. Shear is out of the page, with thermal wind balance implying smaller buoyancy toward positive y. Slantwise ascending parcels acquire positive buoyancy perturbations of the slope of the ascent is less than the slope of constant bouyancy surfaces, which are indicated by the regularly spaced lines.

where W is a constant. Application of the mass continuity equation (8.10) results in

$$v = -(mW/l)\cos(ly + mz)\exp(\nu t) \tag{8.21}$$

for the y component of the wind. Substituting l_C for l from equation (8.18) we find

$$(v, w) = (W/f)(S, f)\cos[m(-fy/S + z)]\exp(\nu t), \tag{8.22}$$

which shows that wave fronts tilt in the positive y direction with slope f/S. The velocity vectors also tilt in the positive y direction with the same slope.

Examination of equation (8.9) shows that surfaces of constant ambient buoyancy tilt in the positive y direction as well, but with slope $fS/N^2 = f/(S\mathrm{J})$. Thus the slope of these ambient isentropic surfaces is equal to the slope of the wave fronts at the threshold for instability $\mathrm{J} = 1$ and is greater than the slope of wave fronts for the unstable case with $\mathrm{J} < 1$. From equation (8.14) we infer that the buoyancy perturbation is

$$b' = \frac{S^2(1 - \mathrm{J})}{\nu} w, \tag{8.23}$$

which means that b' is in phase with w for $\mathrm{J} < 1$. The source of energy for the instability is now evident. Ascending parcels, which would normally acquire negative buoyancy in a stably stratified environment, also move laterally toward a cooler or less buoyant environment. In the unstable case the decrease in environmental buoyancy in the lateral movement outweighs the increase in environmental buoyancy due to vertical motion and the parcel acquires positive buoyancy relative to the environment. Symmetric instability is thus a form of convective

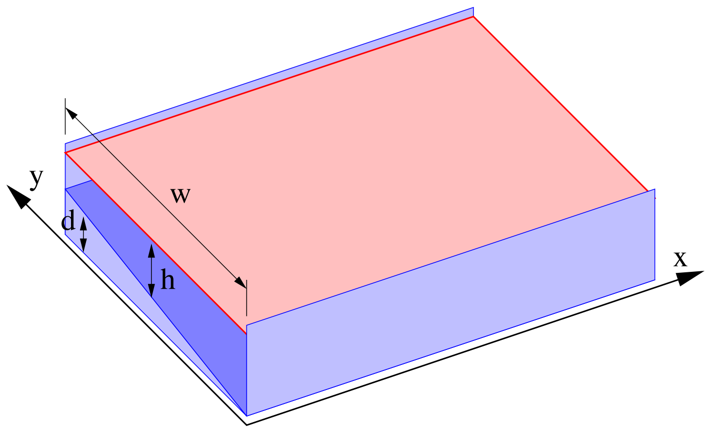

Figure 8.2: Geometry for fluid in a channel of width w with a bottom of variable height $d(y)$.

instability which occurs in strongly sheared environments. For $l = l_C$, equation (8.11) shows that $u' = 0$, which means that the perturbation Coriolis force in the y momentum equation is zero, and $\partial v/\partial t$ is due totally to the kinematic pressure gradient associated with the buoyancy perturbation.

Though earlier work exists, Peter Stone (Stone 1966, 1970, 1972) is primarily responsible for our modern view of symmetric instability. Emanuel (1979, 1982) applied the concept of symmetric instability to mesoscale convective systems and extended the analysis to include moist ascent.

8.2 Rossby waves in the shallow water system

The so-called Rossby wave is a geophysical phenomenon of great importance. It is a nearly balanced flow pattern which occurs in many contexts in oceanic and atmospheric dynamics. Since the flow is approximately balanced, we can use the mathematical apparatus set up to study low Rossby number flow. However, before doing this, we first make some qualitative arguments which reveal the basic physical mechanism of the Rossby wave.

Let us imagine a fluid at rest in a channel with a tilted bottom and level top, as illustrated in figure 8.2. Assuming that the fluid resides in the northern hemisphere of the earth, the Coriolis parameter f is positive and is taken to be constant. The potential vorticity of this fluid varies with y by virtue of the variation in the thickness of the fluid layer with y. Since the fluid layer is thinner for larger y (which we identify as "north"), the potential vorticity, which we recall is equal to $q = f/h$, is larger there.

Figure 8.3 shows how alternating displacements of parcels of fluid north and south from their initial positions affects the potential vorticity distribution. Parcels displaced to the north carry with them potential vorticity values lower than their new environment, and therefore have negative potential vorticity perturbations. The opposite happens with parcels

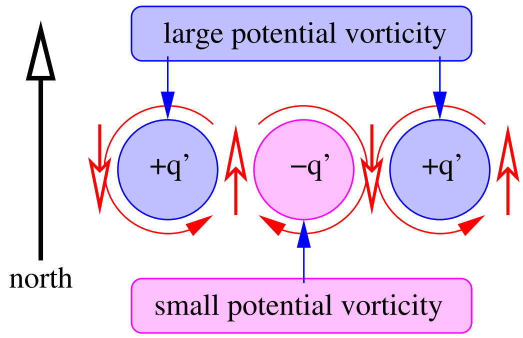

Figure 8.3: Sketch of effect of displacing parcels north and south in a fluid with higher values of potential vorticity to the north.

displaced to the south.

A positive potential vorticity anomaly is associated with positive relative vorticity. A counter-clockwise circulation thus exists about the anomaly in this case. Similarly, a negative anomaly exhibits a clockwise circulation. The net effect of these circulations is to cause northward flow between anomalies where a positive anomaly exists to the west, and southward flow in the opposite case. The northward flow tends to reduce the potential vorticity in the gap between anomalies, whereas the southward flow increases it. Examination of figure 8.3 shows that the net effect of this is to shift all anomalies to the left. We thus have a wave phenomenon in which an east-west train of alternating positive and negative potential vorticity anomalies moves to the west with time. This type of wave is called a *Rossby wave*. Rossby waves play a central role in the large-scale dynamics of the earth's ocean and atmosphere. They depend for their existence on a transverse gradient in potential vorticity. In the present example the potential vorticity gradient is caused by a gradient in the elevation of the lower boundary of the pool of water. Other mechanisms for producing this gradient exist as well, such as the beta effect or a change in the ambient thickness with y due to an ambient wind in the x direction. Meanwhile, let us analyze this situation more quantitatively.

We divide the fractional thickness perturbation into two components, the first representing the north-south thickness variation due to the tilt of the bottom surface, and the second associated with the Rossby wave structure. We assume that the bottom surface depends on y as

$$d/h_0 = \mu y, \tag{8.24}$$

where $y = 0$ at the south wall of the channel. Since the fluid surface at rest must be level, we insist that $h + d = h_0(1 + \eta) + h_0\mu y = h_0$, which means that $\eta = \eta_A = -\mu y$ in the rest case. More generally when there is fluid motion, we postulate that

$$\eta = \eta_A + \eta^*. \tag{8.25}$$

In order to maintain the linearization condition $|\eta| \ll 1$, we must have $\mu w \ll 1$ where w is the channel width, as illustrated in figure 8.2. In the case in which the fluid is in motion, we thus have

$$h + d = h_0(1 + \eta^*), \tag{8.26}$$

which means that the geostrophic velocity is given by

$$v_{gx} = -\frac{gh_0}{f}\frac{\partial \eta^*}{\partial y} \qquad v_{gy} = \frac{gh_0}{f}\frac{\partial \eta^*}{\partial x}. \tag{8.27}$$

At rest the potential vorticity takes the form

$$q = q_A = q_0(1 - \eta_A) = q_0(1 + \mu y). \tag{8.28}$$

In the case with motion we add a potential vorticity perturbation q^* associated with the motion:

$$q_g = q_A + q^*. \tag{8.29}$$

We now substitute equations (8.25), (8.27), and (8.29) into the potential vorticity conservation equation and the inversion equation, both discussed in the previous chapter. We linearize in quantities having to do with motion, i.e., η^*, v_x, v_y, and q^*, and also replace the velocity components with their geostrophic counterparts. Recalling that $q_0 = f/h_0$, the parcel conservation of potential vorticity becomes upon linearization in starred quantities

$$\frac{dq_g}{dt} = \frac{\partial q^*}{\partial t} + v_{gy}\frac{\partial q_0 \mu y}{\partial y} = \frac{\partial q^*}{\partial t} + q_0 \mu f L_R^2 \frac{\partial \eta^*}{\partial x} = 0 \tag{8.30}$$

and we remind ourselves that

$$L_R^2 \nabla^2 \eta^* - \eta^* = q^*/q_0 \tag{8.31}$$

where $L_R^2 = gh_0/f^2$ is the Rossby radius. Note how the terms containing μ, the tilt of the bottom surface, cancel out of equation (8.31), leaving only terms involving motion. The effect of the tilt of the bottom surface enters only into the potential vorticity conservation equation (8.30).

Let us now assume a wave moving in the x direction. Since the fluid is confined to an east-west channel, the y velocity must be zero at the north and south boundaries, which occur at $y = 0, w$. As a result, both η^* and q^* must be zero there as well. Trial solutions which satisfy these boundary conditions are

$$\eta^* = \eta_0^* \sin(\pi y/w) \exp[i(kx - \omega t)] \tag{8.32}$$

and

$$q^* = q_0^* \sin(\pi y/w) \exp[i(kx - \omega t)], \tag{8.33}$$

where η_0^* and q_0^* are constants. Substitution into equations (8.30) and (8.31) yields two linear, homogeneous algebraic equations in two unknowns, which can be represented in matrix form as

$$\begin{pmatrix} -\omega & k\mu f L_R^2 \\ 1 & 1 + L_R^2(k^2 + \pi^2/w^2) \end{pmatrix} \begin{pmatrix} q^* \\ \eta^* \end{pmatrix} = 0. \tag{8.34}$$

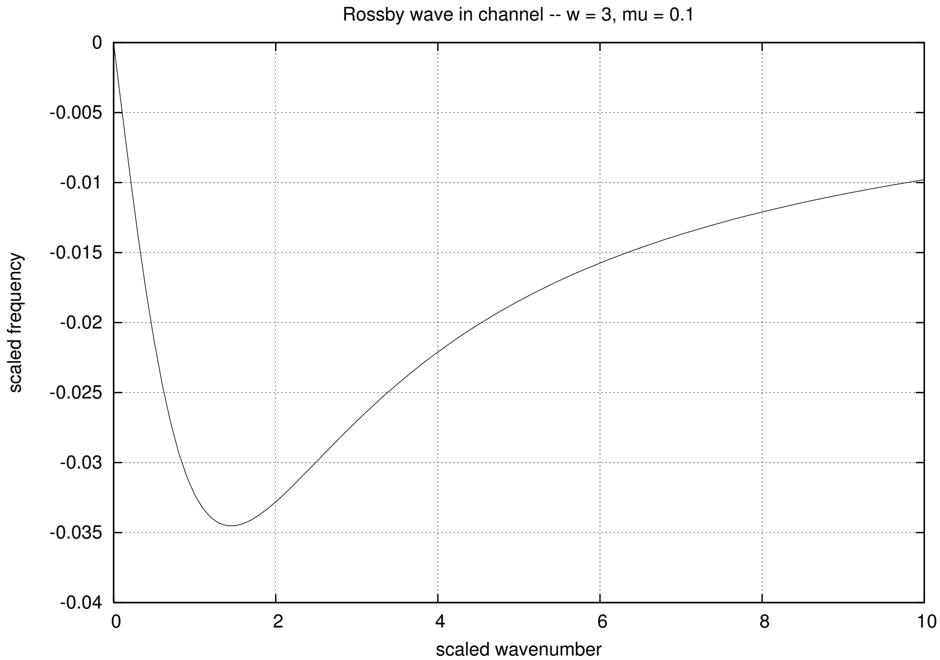

Figure 8.4: Dispersion relation for shallow water Rossby waves in a channel with tilted bottom as represented by equation (8.36) with scaled parameter values $\mu = 0.1$ and $w = 3$.

Setting the determinant of the matrix of coefficients to zero and solving for the frequency ω results in the dispersion relation for Rossby waves

$$\omega = -\frac{kL_R^2 f\mu}{1 + L_R^2(k^2 + \pi^2/w^2)}. \tag{8.35}$$

As predicted by the qualitative arguments outlined above, the phase speed ω/k is negative, i. e., the wave moves in the $-x$ direction, or to the west. Furthermore, the presence of the wavenumber k in the denominator makes this wave dispersive.

It is perhaps easiest to understand this dispersion relation by adjusting the length and time scales so that length is measured in units of the Rossby radius L_R and time is measured in terms of the inverse Coriolis parameter f^{-1}. With this rescaling, the dispersion relation simplifies to

$$\omega = -\frac{\mu k}{1 + k^2 + \pi^2/w^2}. \tag{8.36}$$

Figure 8.4 shows how this dispersion relation behaves for $\mu = 0.1$ and $w = 3$. The magnitude of the frequency peaks for $k = k_c \approx 1.5$. For smaller wavenumbers the group velocity of the wave, $\partial\omega/\partial k$, is negative (i. e., westward) for $k < k_c$ and positive (eastward) for $k > k_c$. Thus, in the short wavelength limit, the group velocity moves in the direction opposite the phase propagation.

8.3 Internal Rossby waves

For stratified flow in the Boussinesq approximation, certain internal Rossby waves are isomorphic to Rossby waves in the shallow water system. More specifically, Rossby waves with vertical wavenumber m in a Boussinesq base state with no vertical or horizontal ambient shear are equivalent to waves in the shallow water case with equivalent depth $N^2/(gm^2)$. Wind shear results in behavior which is not encompassed by the shallow water system.

8.4 Surface Rossby waves

An important type of Rossby wave which arises in a continuously stratified medium, but has no analog in the shallow water equations, is the *surface Rossby wave*. This type of wave occurs when there is a gradient in ambient surface potential temperature. We analyze a pure surface wave on an f-plane with constant potential vorticity in the interior of the flow using quasi-geostrophic dynamics.

The time evolution of the wave is governed by the surface potential temperature evolution equation

$$\frac{\partial \theta_B^*}{\partial t} + u_{gB}\frac{\partial \theta_B^*}{\partial x} + v_{gB}\frac{\partial \theta_B^*}{\partial y} = 0 \tag{8.37}$$

where θ_B^* is the perturbation surface potential temperature. The lower boundary condition relates θ_B^* to the Montgomery potential perturbation M^*,

$$\theta_B^* = \frac{\Gamma_R^2}{N_R^2}\left(\frac{\partial M^*}{\partial \theta}\right)_{B0}, \tag{8.38}$$

and equation (8.37) becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial M^*}{\partial \theta}\right)_{B0} - \frac{1}{f}\left(\frac{\partial M_{B0}^*}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{\partial M^*}{\partial \theta}\right)_{B0} + \frac{1}{f}\left(\frac{\partial M_{B0}^*}{\partial x}\right)\frac{\partial}{\partial y}\left(\frac{\partial M^*}{\partial \theta}\right)_{B0} = 0 \tag{8.39}$$

where we have expressed the geostrophic wind in terms of the perturbation Montgomery potential. Since the geostrophic wind is first order in smallness, we have approximated the wind at the actual boundary θ_B by its value at the the boundary reference level θ_{B0}.

For pure surface waves we take the potential vorticity in the interior of the flow to be constant, which means that advection of potential vorticity in the interior produces no anomalies. Thus $q^* = 0$ and the potential vorticity inversion equation takes the form

$$\frac{1}{f^2}\nabla_h^2 M^* + \frac{\Gamma_R^2}{N_R^2}\frac{\partial^2 M^*}{\partial \theta^2} = 0. \tag{8.40}$$

We express the Montgomery potential anomaly as the sum of a part due to the zonal ambient flow M_Z plus a part due to the surface wave itself M'. A surface temperature gradient implies vertical wind shear in the ambient flow. For generality, we also include a constant ambient wind component U in the x direction, which means that the Montgomery potential anomaly must take the form

$$M^* = M_Z + M' = -f\left[U + \Lambda(\theta - \theta_{B0})\right] y + M'. \tag{8.41}$$

The constant Λ is the wind shear $du_g/d\theta$ in isentropic coordinates. Note that the ambient part of M^* satisfies equation (8.40) by itself, since it is linear in both θ and y. Therefore, substitution of M^* into equation (8.40) reduces to

$$\frac{1}{f^2}\nabla_h^2 M' + \frac{\Gamma_R^2}{N_R^2}\frac{\partial^2 M'}{\partial\theta^2} = 0. \tag{8.42}$$

We seek a wave-like solution in the horizontal, which means that the vertical structure is a linear combination of growing and decaying exponentials in θ. The growing exponential is unphysical, so we assume that $M' = M_S \sin(kx + ly - \omega t)\exp(-mz)$ where M_S is the constant surface amplitude of the wave. From this we easily infer the vertical decay rate of the wave amplitude

$$m = \frac{N_R(k^2 + l^2)^{1/2}}{f\Gamma_R} \tag{8.43}$$

by substitution into equation (8.42). The quantity Γ_R is conversion factor from potential temperature to height, so $m\Gamma_R = N_R(k^2 + l^2)^{1/2}/f$ is the vertical decay length in geometric coordinates. This shows that the vertical extent of the surface wave in geometric coordinates $1/(m\Gamma_R)$ scales with horizontal wavelength $2\pi/(k^2 + l^2)^{1/2}$.

Noting that

$$M^*_{B0} = -fUy + M_S \sin(kx + ly - \omega t) \tag{8.44}$$

and

$$\left(\frac{\partial M^*}{\partial\theta}\right)_{B0} = -f\Lambda y - mM_S \sin(kx + ly - \omega t), \tag{8.45}$$

we substitute these functions into equation (8.39) and assume that the wave amplitude is small compared to the magnitude of the ambient flow. This allows us to linearize in M_S, resulting in a homogeneous equation in this quantity:

$$\left[(\omega - kU)m - k\Lambda\right] M_S = 0. \tag{8.46}$$

Assuming $M_S \neq 0$ requires the expression in square brackets to be zero, which results in the dispersion relation for the surface wave:

$$\omega = kU + \frac{f\Gamma_R k}{N_R(k^2 + l^2)^{1/2}}, \tag{8.47}$$

where we have used equation (8.43) to eliminate m.

The trace speeds of this wave in the x and y directions are

$$u_t = U + \frac{f\Gamma_R}{N_R(k^2 + l^2)^{1/2}} \qquad v_t = \frac{k}{l} u_t. \tag{8.48}$$

Thus, the phase propagation relative to the surface x wind U is toward positive x irrespective of the direction of the wave vector (k, l). The components of group velocity are

$$u_g = U + \frac{f\Gamma_R l^2}{N_R(k^2 + l^2)^{3/2}} \qquad v_g = -\frac{f\Gamma_R kl}{N_R(k^2 + l^2)^{3/2}}, \tag{8.49}$$

which shows that the phase progression in the y direction is opposite the group velocity, a behavior reminiscent of the vertical propagation of gravity waves. Thus, surface Rossby waves are both dispersive and anisotropic. Note that if $l = 0$, the x component of the group velocity equals the surface wind speed in that direction.

The plane wave solution is technically invalid, as it extends to infinity in all directions; the base state violates the initial assumptions of quasi-geostrophic theory for large values of $|y|$. However, if the plane wave solutions are used to construct a wave packet with sufficiently limited horizontal dimensions, then the result can be considered valid. Alternatively, the solution could be confined to a channel with free-slip boundaries, as in the shallow water Rossby wave considered above.

8.5 References

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8.6 Questions and problems

1. Show that the ambient environment has negative potential vorticity (assuming $f > 0$) if it is unstable to symmetric instability.

2. Redo the analysis of the shallow water Rossby wave with zero terrain but non-zero beta effect.

3. In zonal symmetry ($\partial/\partial x = 0$) on an f-plane, show that the quantity $M \equiv u - fy$ is conserved by parcels. (This works for the shallow water equations and all forms of the stably stratified three-dimensional equations, so take your pick.) Hint: Use $v = dy/dt$.

4. Repeat the above analysis on an equatorial beta-plane, deriving the form of M in this case.

5. Consider a shallow water flow in an east-west channel of north-south width w ($0 \leq y \leq w$) with flat bottom and with a flow moving uniformly in geostrophic balance to the east at speed U. Assume constant f.

   (a) For steady flow, show that the fractional thickness perturbation takes the form $\eta_A = -Uy/(fL_R^2) \equiv -\mu y$. We assume that $\mu w \ll 1$.

   (b) Show that the potential vorticity in this steady flow situation is $q_A(y) = q_0(1+\mu y)$.

   (c) Assume a wave of the form $(q^*, \eta^*) \propto \sin(\pi y/w) \exp[i(kx - \omega t)]$ on this basic flow, where $\eta = \eta_A + \eta^* = -\mu y + \eta^*$, and where the potential vorticity $q = q_A + q^* = q_0(1 + \mu y) + q^*$, and find the dispersion relation $\omega = \omega(k)$. Hint: Linearize the potential vorticity evolution equation about a state of uniform motion.

   (d) Compute the geostrophic flow velocity due to the wave and the total flow velocity.