Chapter 7

Quasi-Geostrophic Theory

In the last chapter we defined the potential vorticity in a number of contexts and showed that in the absence of heating and friction it is a quantity conserved by fluid parcels. In this chapter we show that the potential vorticity also carries complete information about the balanced part of the flow as well as additional useful information about the unbalanced component. By "balance", we refer to geostrophic balance, though many of the arguments carry over to more sophisticated forms such as gradient and nonlinear balance. We confine our discussion here to geostrophic balance. In the geostrophic case the balanced part of the flow is simply the geostrophic wind. The unbalanced part is the total wind minus the geostrophic wind and is called the *ageostrophic* wind.

The process by which we extract information from the potential vorticity is called *potential vorticity inversion*, first popularized by Hoskins, McIntyre, and Robertson (1985). It is simplest to understand inversion in the context of the shallow water and isentropic coordinate systems. We discuss these cases in turn. In both cases inversion works best when the ageostrophic wind is small. Since the ageostrophic wind is associated with the acceleration term in the horizontal momentum equation, we can evaluate the relative magnitude of the ageostrophic component via a scale analysis on the momentum equation. Taking the shallow water momentum equation

$$\frac{d\boldsymbol{v}}{dt} + g\boldsymbol{\nabla}(h+d) + f\boldsymbol{k}\times\boldsymbol{v} = 0 \tag{7.1}$$

as an example, we note that the acceleration term scales as V/T where V is a characteristic velocity and T is a characteristic time scale of the disturbance in question. If the acceleration term is small compared to the other terms in equation (7.1), then the height gradient and the Coriolis terms are roughly the same order of magnitude, and we can perform a scale analysis on either one for purposes of comparison with the acceleration term. The Coriolis term scales as fV. The ratio of the estimated acceleration and Coriolis terms is called the *Rossby number*:

$$\mathrm{Ro} = \frac{1}{fT}. \tag{7.2}$$

The acceleration is therefore small compared to the Coriolis force when $\mathrm{Ro} \ll 1$, or when $T \gg 1/f$. Potential vorticity inversion using geostrophic balance is therefore confined to disturbances with time scales greater than the rotation period of the earth. This encompasses most large-scale meteorological disturbances.

Charney (1948) was responsible for the systematic development of the above ideas in the so-called *quasi-geostrophic* theory, using the method of scale analysis on the full governing equations. The name comes from the fact that it works when fluid motions are close to, but not exactly in geostrophic balance.

We take an alternate approach to Charney's theory by applying what we have learned about potential vorticity. The basic idea of quasi-geostrophic theory (and other quasi-balanced theories) is that the potential vorticity is inverted to obtain the geostrophic velocity and the associated mass field. The mass continuity equation is then used to infer the "slaved" part of the ageostrophic flow. Given this information, the parcel conservation of potential vorticity allows the potential vorticity field to be stepped forward in time and the process is repeated. In this way the time evolution of the flow can be computed.

We first examine quasi-geostrophic theory in the context of the shallow water equations and then extend our analysis to the three-dimensional case using the isentropic coordinate equations. Common to both cases, quasi-geostrophic theory assumes the perturbations to the layer thickness (shallow water case) or the isentropic density (isentropic case) are assumed to be small compared to the base state values. However, velocities cannot in general assumed to be small enough to neglect advection terms in material derivatives, though we shall continue to impose this condition in further linearizations.

Quasi-geostrophy advects potential vorticity and other quantities with the geostrophic rather than the total wind. This eliminates the physics of fronts from the theory. The more sophisticated *semi-geostrophic* theory includes the ageostrophic part of the wind in advection. See Hoskins (1975) for an exposition of this theory.

7.1 Shallow water case

The shallow water potential vorticity is

$$q = \frac{\zeta_a}{h} \tag{7.3}$$

where the absolute vorticity is

$$\zeta_a = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \tag{7.4}$$

and h is the thickness of the fluid layer. The geostrophic absolute vorticity ζ_{ag} is obtained by substituting the geostrophic wind

$$u_g = -\frac{g}{f}\frac{\partial(h+d)}{\partial y} \qquad v_g = \frac{g}{f}\frac{\partial(h+d)}{\partial x} \tag{7.5}$$

for the real wind, resulting in

$$\zeta_{ag} = hq_g = \frac{\partial}{\partial x}\left(\frac{g}{f}\frac{\partial(h+d)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{g}{f}\frac{\partial(h+d)}{\partial y}\right) + f. \tag{7.6}$$

The terrain elevation is $d(x, y)$. This is a linear partial differential equation for the layer thickness in terms of the geostrophic potential vorticity q_g which is elliptic as long as f and q_g are either uniformly positive or negative. It thus has a unique solution as long as appropriate boundary conditions are satisfied.

The special case of small amplitude disturbances on a beta-plane helps illustrate what is going on here. In this case we make the usual assumption that $h = h_0(1+\eta)$ where $|\eta| \ll 1$. On a beta-plane we let $f = f_0 + \beta y$ where $|\beta y| \ll f_0$. In addition we assume that the relative vorticity is much smaller than f_0 Linearizing the potential vorticity in this case, we find

$$q_g \approx \frac{f_0}{h_0}\left[1 + \frac{\beta y}{f_0} + \frac{1}{f_0}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) - \eta\right] = q_A + q_0\left(L_R^2\nabla^2\eta^* - \eta^*\right) \tag{7.7}$$

where

$$q_A = q_0\left[1 + \beta y/f_0 + (d/h_0)\right] \tag{7.8}$$

is the base state potential vorticity associated with the Coriolis parameter and variations of terrain elevation. The quantity $\eta^* = \eta + d/h_0$ is the part of the fractional thickness perturbation associated with motion, since as equation (7.5) shows, constant η^* corresponds to zero geostrophic wind. The quantity $L_R^2 = gh_0/f_0^2$ is the square of the *Rossby radius*, a characteristic length scale for nearly geostrophic, shallow water disturbances away from the equator and $q_0 = f_0/h_0$ is the *planetary potential vorticity.* This equation may be rewritten in a more convenient form

$$L_R^2\nabla^2\eta^* - \eta^* = q^*/q_0 \tag{7.9}$$

where $q^* = q_g - q_A$ is the part of the potential vorticity associated with motion. The geostrophic wind components written in terms of η^* are

$$u_g = -f_0 L_R^2\frac{\partial\eta^*}{\partial y} \qquad v_g = f_0 L_R^2\frac{\partial\eta^*}{\partial x}. \tag{7.10}$$

The mass continuity equation has not yet been invoked in this analysis, and it contains useful information about the ageostrophic flow. We write this as

$$\frac{\partial\eta}{\partial t} + \boldsymbol{v}_g\cdot\boldsymbol{\nabla}\eta + \boldsymbol{\nabla}\cdot\boldsymbol{v}_a = 0, \tag{7.11}$$

noting that $\boldsymbol{\nabla}\cdot\boldsymbol{v}_g = 0$, which means that $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{v}_a$ where $\boldsymbol{v}_a = \boldsymbol{v} - \boldsymbol{v}_g$ is the ageostrophic wind. The total wind is replaced by the geostrophic wind in the advection term since $|\boldsymbol{v}_g| \gg |\boldsymbol{v}_a|$ for low Rossby number flow. Equation (7.11) imposes a constraint on the slaved ageostrophic wind associated with balanced motions. It does not reveal anything about unbalanced phenomena such as inertia-gravity waves. Furthermore, the constraint acts only on the irrotational part of the ageostrophic flow. Since equation (7.9) diagnoses η in the past as well as in the present time, the time derivative of η can be taken as known. The irrotational part of the ageostrophic flow (assumed here to be the only part of interest) can be represented by a velocity potential χ:

$$\boldsymbol{v}_a = -\boldsymbol{\nabla}\chi. \tag{7.12}$$

Substituting this into equation (7.11) yields a Poisson equation for the velocity potential in terms of η and the geostrophic wind:

$$\nabla^2\chi = \frac{\partial \eta}{\partial t} + \boldsymbol{v}_g \cdot \boldsymbol{\nabla}\eta. \tag{7.13}$$

In summary, once the distribution of potential vorticity is known in a domain confined to one side of the equator (so that the Coriolis parameter does not change sign within the domain), the fluid layer thickness, the geostrophic velocity field, and a slaved ageostrophic velocity can be derived. Though this is hard to prove rigorously, experience shows that a flow which is initially far from balance tends to evolve toward a quasi-balanced state, potentially with unbalanced inertia-gravity waves generating oscillations about this state.

Once the velocity is known, we evolve the potential vorticity. This is done with the potential vorticity conservation equation

$$\frac{\partial q_g}{\partial t} + \boldsymbol{v}_g \cdot \boldsymbol{\nabla} q_g = 0. \tag{7.14}$$

Note that the total potential vorticity q_g enters into this equation. The part associated with motion can be extracted from the total: $q^* = q_g - q_A$. Once q^* is known at the new time, the inversion process can be repeated.

7.2 Anelastic isentropic case

We now extend the analysis of potential vorticity inversion to the full three-dimensional case, making the isentropic anelastic approximation to avoid non-essential mathematical complexity. The advantage over analyzing potential vorticity inversion in geometric or pressure coordinates is that the horizontal components of vorticity vanish from the definition of potential vorticity. In fact, the potential vorticity in isentropic coordinates looks a great deal like the shallow water version:

$$q = \frac{\zeta_a}{\sigma} = \frac{1}{\sigma}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) \tag{7.15}$$

where the fractional thickness perturbation is replaced by σ, the density in isentropic coordinates. The quantity ζ_a in this case is not the vertical component of absolute vorticity in geometric coordinates, but the component normal to isentropic (constant potential temperature) surfaces. Of course, ζ_a is the vertical component of vorticity in isentropic coordinate space.

The geostrophic velocity components in isentropic coordinates are

$$u_g = -\frac{1}{f}\frac{\partial M}{\partial y} \qquad v_g = \frac{1}{f}\frac{\partial M}{\partial x} \tag{7.16}$$

where we recall that M is the Montgomery potential. The Montgomery potential is related to the isentropic density in the anelastic case by

$$\frac{\partial^2 M}{\partial \theta^2} = -\frac{N^2}{\sigma_0 \Gamma^2}\sigma \tag{7.17}$$

where $N(\theta)$ is the base state profile of Brunt-Väisälä frequency, $\Gamma(\theta) = (dz_0/d\theta)^{-1}$, and where we have anticipated the need for a base state density profile $\sigma_0(\theta)$. Combining equations (7.15)-(7.17), the geostrophic approximation to the potential vorticity equation in this case is

$$f + \frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial M}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial M}{\partial y}\right) + \frac{\sigma_0 \Gamma^2 q_g}{N^2}\frac{\partial^2 M}{\partial \theta^2} = 0, \tag{7.18}$$

where q_g is the quasi-geostrophic potential vorticity. This is a linear, second order partial differential equation in M which is elliptic under conditions similar to those which pertain in the shallow water case.

Assuming that $f = f_0 + \beta y$, $\sigma = \sigma_0(\theta) + \sigma^*$, and $M = M_0(\theta) + M^*$, with $|\beta y| \ll f_0$, $|\sigma^*| \ll \sigma_0$, and $|M^*| \ll M_0$, we linearize equation (7.18). We also assume that σ_0 and M_0 satisfy equation (7.17) so that the base state terms cancel in equation (7.18). Under these conditions, equation (7.18) becomes

$$\frac{1}{f_0^2}\nabla_h^2 M^* + \frac{\Gamma^2}{N^2}\frac{\partial^2 M^*}{\partial \theta^2} = \frac{q^*}{q_0} \tag{7.19}$$

where ∇_h^2 is the two-dimensional, horizontal (in isentropic coordinate space) Laplacian. As in the shallow water case, the geostrophic potential vorticity is split into a base state part and a part having to do with motion, $q_g = q_A + q^*$. In the present case the base state part is

$$q_A = q_0(1 + \beta y/f_0) \tag{7.20}$$

where the planetary potential vorticity is $q_0 = f_0/\sigma_0$.

The three-dimensional treatment of potential vorticity inversion exhibits one complication which does not exist in the shallow water case. Equations (7.18) and (7.19) require a lower boundary condition for their solution. A physically sensible mixed boundary condition arises from the geopotential diagnostic equation:

$$\Phi = M - \theta\frac{\partial M}{\partial \theta}. \tag{7.21}$$

Evaluating this at the lower boundary θ_B results in

$$\Phi_B = M_B - \theta_B \left(\frac{\partial M}{\partial \theta} \right)_B \tag{7.22}$$

where the subscripted B indicates evaluation at the bottom boundary θ_B. The geopotential at the lower boundary is just $\Phi_B = gd(x, y)$ where d is the terrain elevation.

Evaluation of the lower boundary condition using equation (7.22) directly is awkward, since the potential temperature of the lower boundary varies with position and time. However, we can define a fixed reference level θ_{B0} such that $\theta_B = \theta_{B0} + \theta_B^*$ and make a first-order Taylor series expansion in θ to relate Φ_B to quantities at this level:

$$\Phi_B \approx \Phi_{B0} + \left(\frac{\partial \Phi_0}{\partial \theta} \right)_{B0} \theta_B^* \approx M_{B0} - \theta_{B0} \left(\frac{\partial M}{\partial \theta} \right)_{B0} + \left(\frac{d\Phi_0}{d\theta} \right)_{B0} \theta_B^*. \tag{7.23}$$

We have split the geopotential into a base state profile and a perturbation $\Phi = \Phi_0 + \Phi^*$ and have dropped Φ^* in the term containing θ_B^*, consistent with linearization. Equation (7.23) must be satisfied by the base state, resulting in

$$\Phi_{0B} = M_{0B0} - \theta_{B0} \left(\frac{dM_0}{d\theta} \right)_{B0}. \tag{7.24}$$

Subtracting equation (7.24) from equation (7.23) produces our linearized lower boundary condition:

$$\Phi_B^* = M_{B0}^* - \theta_{B0} \left(\frac{\partial M^*}{\partial \theta} \right)_{B0} + \left(\frac{d\Phi_A}{d\theta} \right)_{B0} \theta_B^*. \tag{7.25}$$

This boundary condition appears to be rather complicated. However, note that it solves the formidable problem of the variable and dynamic lower boundary characteristic of isentropic coordinates by approximating the actual boundary condition with a condition which is applied, not at the actual lower boundary, but at a constant reference level $\theta = \theta_{B0}$. This works as long as terrain elevation and surface potential temperature variations are small enough to justify the linearization.

The slaved ageostrophic wind $\boldsymbol{v}_a = -\boldsymbol{\nabla}_h \chi$ is obtained from a velocity potential χ, which is governed by the mass continuity equation. The mass continuity equation linearized in σ becomes

$$\nabla_h^2 \chi = \frac{1}{\sigma_0} \left(\frac{\partial \sigma^*}{\partial t} + \boldsymbol{v}_g \cdot \boldsymbol{\nabla}_h \sigma^* \right) \tag{7.26}$$

where σ^* is derived from the Montgomery potential using equation (7.17):

$$\sigma^* = -\frac{\sigma_0 \Gamma^2}{N^2} \frac{\partial^2 M^*}{\partial \theta^2}. \tag{7.27}$$

We set the heat source term S to zero and treat heating effects later.

As in the quasi-geostrophic shallow water case, the potential vorticity is advected only by the geostrophic wind

$$\frac{\partial q_g}{\partial t} + \boldsymbol{v}_g \cdot \boldsymbol{\nabla}_h q_g = 0 \tag{7.28}$$

and the result is used to derive $q^* = q_g - q_A$. The surface temperature perturbation obeys

$$\frac{\partial \theta_B^*}{\partial t} + \boldsymbol{v}_{gB0} \cdot \boldsymbol{\nabla}_h \theta_B^* = 0 \tag{7.29}$$

where $\boldsymbol{v}_{gB0}$ is the geostrophic wind there. We approximate both of these by their values at the reference surface θ_{B0}.

7.3 Boussinesq isentropic coordinate case

In the Boussinesq case $\Gamma \rightarrow \Gamma_R$ and $N \rightarrow N_R$, where Γ_R, N_R, and σ_0 are assumed constant. Thus, (7.19) becomes

$$\frac{1}{f_0^2}\nabla_h^2 M^* + \frac{\Gamma_R^2}{N_R^2}\frac{\partial^2 M^*}{\partial \theta^2} = \frac{q^*}{q_0}. \tag{7.30}$$

The relationship between Montgomery potential and isentropic density simplifies to

$$\frac{\partial^2 M}{\partial \theta^2} = -\frac{N_R^2 \sigma}{\Gamma_R^2 \sigma_0} \tag{7.31}$$

and the geopotential diagnostic becomes

$$\Phi = M_R - \theta_R \frac{\partial M}{\partial \theta} \tag{7.32}$$

where M_R and θ_R are constant reference values of M and θ. Equation (7.32) derives from the fact that the Boussinesq approximation is technically only valid in atmospheric layers much thinner than the scale height, which means that variations in M and θ are small, but the ratio of these variations as expressed in the derivative is large. Since the Exner function $\Pi = \partial M/\partial \theta$, we can pick a reference value of this variable characteristic of conditions near the earth's surface, $\Pi_R = C_p$. If we insist that $\Phi = 0$ there, then the reference value of the Montgomery potential becomes $M_R = \theta_R \Pi_R = \theta_R C_p$. Thus, a simple relationship between geopotential, Montgomery potential, and Exner function exists in the Boussinesq case,

$$\Phi = \theta_R \left(C_p - \frac{\partial M}{\partial \theta} \right) = \theta_R (C_p - \Pi). \tag{7.33}$$

Equation (7.23) for the surface boundary condition becomes

$$\Phi_B = \Phi_{B0} + \left(\frac{\partial \Phi_0}{\partial \theta} \right)_{B0} \theta_B^* = \theta_R \left[C_p - \left(\frac{\partial M}{\partial \theta} \right)_{B0} \right] + \frac{\theta_R N_R^2}{\Gamma_R^2} \theta_B^*, \tag{7.34}$$

where we have taken advantage of the fact that $d\Phi_A/d\theta = -\theta_R(d^2M_A/d\theta^2) = \theta_R N_R^2/\Gamma_R^2$ in the Boussinesq case. For the base state this becomes

$$\Phi_{AB} = \theta_R \left[C_p - \left(\frac{\partial M_0}{\partial \theta} \right)_{B0} \right] \tag{7.35}$$

and the condition on perturbation quantities comes from subtracting equation (7.35) from equation (7.34):

$$\Phi_B^* = -\theta_R \left(\frac{\partial M^*}{\partial \theta} \right)_{B0} + \frac{\theta_R N_R^2}{\Gamma_R^2} \theta_B^*. \tag{7.36}$$

The Boussinesq case is the same as the anelastic isentropic case in all other respects.

7.4 References

Charney J. G., 1948: On the scale of atmospheric motion. *Geofys. Publ. Oslo*, **17(2)**, 1-17. This presents the first systematic development of quasi-geostrophic theory. The paper is available in Geoff Vallis's reprint archive (see the main web page for the course).

Hoskins, B. J., 1975: The geostrophic momentum approximation and the semi-geostrophic equations. *J. Atmos. Sci.*, **32**, 233-243. This is a good place to start on semi-geostrophic theory.

Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877-946. This introduces the idea of potential vorticity inversion.

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics.* Cambridge University Press, 745 pp. Isentropic coordinates are developed in chapter 3. The circulation theorem and potential vorticity are discussed in chapter 4. Quasi-geostrophic theory is presented in chapter 5.

7.5 Questions and problems

1. Imagine a point vortex in purely two-dimensional, horizontal flow, where the vorticity field is given by $\zeta_z = C\delta(x)\delta(y)$ with C a constant equal to the strength of the vortex. The quantity $\delta()$ is the Dirac delta function. Solve for the horizontal velocity on an infinite domain. Hint: Use cylindrical symmetry and the Kelvin circulation theorem applied to a circular loop centered on the vortex to obtain the velocity field. If you have experience with electromagnetism, think of the problem of the magnetic field surrounding an infinite wire carrying a current.

2. Consider a two-dimensional flow which is stationary except for the flows associated with two point vortices of equal but opposite strength $\pm C$ separated by a distance d. Describe the speed and direction of motion of the two vortices.

3. Repeat the above problem for the case in which the two vortices have strength of the same sign and magnitude.

4. Given the potential vorticity distribution $q_g = q_0 [1 + \epsilon \sin(kx)]$ in the shallow water case on an f-plane with no terrain:

   (a) Invert to obtain the fractional thickness perturbation η and the geostrophic velocity $\boldsymbol{v}_g$.

   (b) From the geostrophic velocity, obtain the relative vorticity ζ.

   (c) Sketch plots of the maximum amplitude of η/ϵ and $\zeta/(\epsilon f)$ vs kL_R and comment on the relative importance of thickness perturbations to vorticity in small-scale ($kL_R \gg 1$) and large-scale ($kL_R \ll 1$) disturbances.

5. Consider a distribution of potential vorticity in the shallow water case of the form $q_g = q_0 [1 + \epsilon\delta(x)]$ where ϵ is a constant. Assume no terrain and constant f.

   (a) Invert to obtain $\eta(x)$ and $\boldsymbol{v}_g(x)$. Sketch these variables as functions of x/L_R.

   (b) Assume now that $\epsilon = Ct$ where C is a constant, i.e., ϵ increases linearly with time. The above solution now makes η time-dependent. From η, compute the velocity potential χ and the slaved ageostrophic wind $\boldsymbol{v}_a$.

   (c) Comment on the role of $\boldsymbol{v}_a$ in maintaining mass continuity.

6. In the isentropic Boussinesq case with constant f:

   (a) Rescale the θ coordinate so that $d\theta = (f\Gamma_R/N_R)d\xi$ in the potential vorticity inversion equation.

   (b) Ignoring boundary conditions, obtain M' as a function of $r = (x^2 + y^2 + \xi^2)^{1/2}$ assuming that $q_g'/q_0 = C\delta(x)\delta(y)\delta(\xi)$. Hint: Think Coulomb's law.

   (c) Compute the geostrophic wind $\boldsymbol{v}_g$. Comment on the dependence of $|\boldsymbol{v}_g|$ on r at the level $\xi = 0$; compare with that of a vortex in two-dimensional flow.

   (d) Compute the geopotential height perturbation. Is the air warmer or colder than the base state above the potential vorticity anomaly? Take C and f to be positive.