Chapter 6

Circulation Theorem and Potential Vorticity

Many phenomena in geophysical fluid dynamics have characteristic time scales that are significantly greater than the inverse of the Coriolis parameter $1/f$. On such time scales the horizontal velocities are close their geostrophically balanced values. In this case approximations can be made which completely eliminate free gravity waves from the equations governing geophysical fluid dynamics. The most compact and logical way to present these approximations is in the context of a quantity called the *potential vorticity*. The potential vorticity in turn is related to the Kelvin circulation theorem expressed in a rotating environment. Potential vorticity is a quantity related to the vorticity and stratification of the fluid under consideration. In the absence of friction, heating, and external forces, the potential vorticity is conserved by parcels. The derivation of this conservation condition suggests the use of the potential temperature as an alternate vertical coordinate, which leads to concept of the *isentropic coordinate system*.

6.1 Kelvin circulation theorem

We start by recalling the definition of vorticity $\boldsymbol{\zeta} = \boldsymbol{\nabla} \times \boldsymbol{v}$. The vorticity is related to a quantity called the *circulation*, defined as the closed line integral of the fluid velocity component parallel to the path:

$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{l} = \int \boldsymbol{\nabla} \times \boldsymbol{v} \cdot \boldsymbol{n} dA = \int \boldsymbol{\zeta} \cdot \boldsymbol{n} dA. \tag{6.1}$$

The second form of the circulation involving the vorticity is obtained using Stokes' theorem, with $\boldsymbol{n}$ being the unit normal to the area element dA. The area integral is over the region bounded by the circulation path. Figure 6.1 illustrates the circulation loop.

Of particular interest is the circulation loop which moves and deforms with the fluid flow.

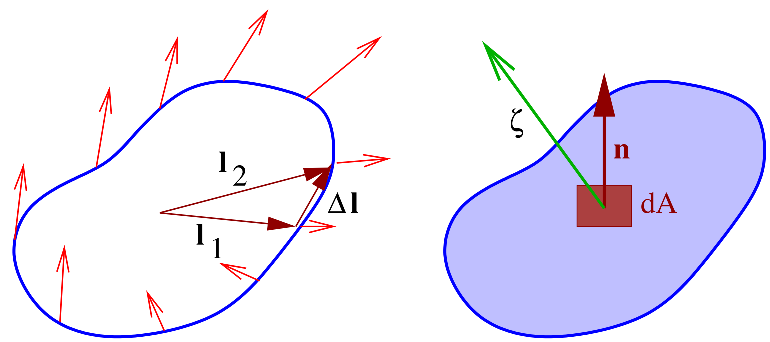

Figure 6.1: Sketch of a circulation loop which advects with the fluid flow, symbolized by the arrows on the left. Stokes' theorem relates the circulation to the integral of the component of vorticity normal to the area bounded by the loop, as shown on the right.

The area, shape, and orientation of this loop evolve with time. However, the time rate of change of the circulation around such a loop obeys a surprisingly simple law, as we now show.

We wish to take the time derivative of Γ. However, the fact that the circulation loop evolves with time complicates this calculation. It is simplest to write the circulation integral in finite sum form in which $d\boldsymbol{l} \to \Delta\boldsymbol{l}_i = \boldsymbol{l}_{i+1} - \boldsymbol{l}_i$ as illustrated in the left panel of figure 6.1:

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\sum \boldsymbol{v}_i \cdot (\boldsymbol{l}_{i+1} - \boldsymbol{l}_i) = \sum \frac{d\boldsymbol{v}_i}{dt} \cdot \Delta\boldsymbol{l}_i + \sum \boldsymbol{v}_i \cdot \Delta\boldsymbol{v}_i, \tag{6.2}$$

where $\boldsymbol{v}_i = d\boldsymbol{l}_i/dt$. We then revert to integral forms and note further that $\boldsymbol{v} \cdot d\boldsymbol{v} = d(v^2/2)$, which results in

$$\frac{d\Gamma}{dt} = \oint \frac{d\boldsymbol{v}}{dt} \cdot d\boldsymbol{l} + \oint d(v^2/2). \tag{6.3}$$

The second term on the right is the line integral of a perfect differential over a closed path and is therefore zero.

The total time derivative of velocity can be eliminated using the momentum equation:

$$\frac{d\boldsymbol{v}}{dt} + \theta\boldsymbol{\nabla}\Pi + \boldsymbol{\nabla}\Phi + 2\boldsymbol{\Omega} \times \boldsymbol{v} = 0. \tag{6.4}$$

We work for now in an inertial reference frame in which $\boldsymbol{\Omega} = 0$ and introduce a rotating frame at a later stage. In this case equation (6.3) becomes

$$\frac{d\Gamma}{dt} = -\oint (\theta\boldsymbol{\nabla}\Pi + \boldsymbol{\nabla}\Phi) \cdot d\boldsymbol{l} = -\oint (\theta\boldsymbol{\nabla}\Pi \cdot d\boldsymbol{l} + d\Phi). \tag{6.5}$$

The integral of a perfect differential around a closed loop is zero, so we arrive at the *Kelvin circulation theorem*:

$$\frac{d\Gamma}{dt} = -\oint \theta\boldsymbol{\nabla}\Pi \cdot d\boldsymbol{l}. \tag{6.6}$$

There are two cases for which the right side of equation (6.6) is zero. If the circulation loop is contained within a surface of constant Exner function (or constant pressure if the $\rho^{-1}\boldsymbol{\nabla}p$ form is used), then $\boldsymbol{\nabla}\Pi \cdot d\boldsymbol{l} = 0$. This case is not particularly useful, as there is no guarantee that the circulation loop will remain in this surface. The alternative case occurs when the loop is contained in a surface of constant potential temperature. In this case $\theta\boldsymbol{\nabla}\Pi \cdot d\boldsymbol{l} = \boldsymbol{\nabla}(\theta\Pi) \cdot d\boldsymbol{l} = d(\theta\Pi)$ and the integrand on the right side of equation (6.6) becomes a perfect differental, and the integral is zero. If no heating occurs, then potential temperature is conserved by parcels, and the circulation loop, which we recall moves and deforms with the flow, will remain in this surface. Therefore in this case Γ is constant with time.

In geophysical fluid dynamics we always use the circulation as computed in an inertial reference frame. However, we often have to compute the circulation from the fluid velocity in the rotating frame of the earth. The velocity in the inertial frame $\boldsymbol{v}_I$ is related to the velocity in the rotating frame $\boldsymbol{v}$ by

$$\boldsymbol{v}_I = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r}, \tag{6.7}$$

where $\boldsymbol{\Omega}$ is the rotation vector of the earth and $\boldsymbol{r}$ is the position vector relative to the center of the earth. The circulation thus becomes

$$\Gamma = \oint \boldsymbol{v}_I \cdot d\boldsymbol{l} = \oint \boldsymbol{v} \cdot d\boldsymbol{l} + \int [\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r})] \cdot \boldsymbol{n} dA, \tag{6.8}$$

where we have used Stokes' theorem to convert the second line integral into an area integral bounded by the circulation loop. A vector identity can be used to reduce the last term: $\boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega}(\boldsymbol{\nabla} \cdot \boldsymbol{r}) - \boldsymbol{\Omega} \cdot \boldsymbol{\nabla}\boldsymbol{r} = 3\boldsymbol{\Omega} - \boldsymbol{\Omega} = 2\boldsymbol{\Omega}$. Substituting this into equation (6.8) results in

$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{l} + \int 2\boldsymbol{\Omega} \cdot \boldsymbol{n} dA = \int (\boldsymbol{\nabla} \times \boldsymbol{v} + 2\boldsymbol{\Omega}) \cdot \boldsymbol{n} dA = \int \boldsymbol{\zeta}_a \cdot \boldsymbol{n} dA \tag{6.9}$$

where we recognize

$$\boldsymbol{\zeta}_a = \nabla \times \boldsymbol{v}_I = \boldsymbol{\nabla} \times \boldsymbol{v} + 2\boldsymbol{\Omega} \tag{6.10}$$

as the absolute vorticity.

6.2 Potential vorticity

The Kelvin circulation theorem encompasses the same physics as does the vorticity equation, but in a form which is perhaps somewhat easier to understand. Considering a circulation loop of small size embedded in an isentropic surface, or a horizontally oriented circulation loop in the shallow water case, we note that $\Gamma \approx \zeta_a A$ where ζ_a is the component of absolute vorticity normal to the loop and A is the area of the loop. Since Γ is constant, increases in the area A as a result of deformation of the flow are accompanied by decreases in ζ_a

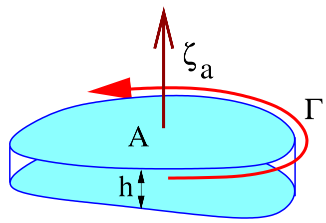

Figure 6.2: Segment of shallow water flow to which we apply the circulation theorem. The two surfaces of area A indicate the upper and lower bounds of the flow. The region between them has volume Ah. The vertical component of the absolute vorticity is ζ_a.

and vice versa. Since the loop moves with the parcel in which it is embedded, we see that the circulation around a parcel is a quantity that is conserved by the parcel. Aside from a multiplicative constant, this quantity is called the *potential vorticity*. It can be thought of as the component of the absolute vorticity normal to the circulation loop which occurs when the loop is deformed so as to have unit area.

We now determine the form of the potential vorticity in the shallow water, incompressible fluid, and ideal gas cases.

6.2.1 Shallow water case

In the shallow water case we consider a vertical cylinder extending through the full depth of the fluid. In the shallow water approximation the flow doesn't vary with depth, so the circulation around this parcel of fluid is independent of depth. Applying the circulation theorem to a (nearly) horizontal loop bounding this fluid parcel as shown in figure 6.2, we conclude from the Kelvin theorem that the circulation around the segment is conserved. We can write the circulation as

$$\Gamma = \zeta_a A = \frac{\zeta_a}{h}(Ah). \tag{6.11}$$

The quantity Ah is the volume of the fluid parcel. As the fluid is incompressible, the volume of this parcel does not change with time. Since the circulation around the parcel is conserved, the variable

$$q \equiv \frac{\zeta_a}{h} = \frac{1}{h}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) \tag{6.12}$$

which is known as the potential vorticity, is also conserved by parcels, i.e.,

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0. \tag{6.13}$$

The vertical component of absolute vorticity increases as the area of the loop decreases. However, a decrease in the loop area implies an increase in the thickness h of the fluid layer. This increase is in proportion to the increase in absolute vorticity, which is why the ratio of the two quantities in a parcel stays the same.

6.2.2 Incompressible ocean

We approximate the ocean as incompressible, but with spatially variable density. We therefore consider a small parcel in a stratified ocean of horizontal area A and vertical thickness h, bounded above and below by constant density surfaces ρ_2 and ρ_1 with $\rho_2 < \rho_1$. The circulation in a loop contained in a constant density surface and bounded by this parcel is

$$\Gamma = \zeta_a A = \zeta_a \left(\frac{\rho_2 - \rho_1}{h\rho}\right)\left(\frac{hA\rho}{\rho_2 - \rho_1}\right) = \frac{\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\rho}{\rho}\left(\frac{hA\rho}{\rho_2 - \rho_1}\right), \tag{6.14}$$

where $\rho = (\rho_1 + \rho_2)/2$ is the mean parcel density. The quantity $hA\rho$ is the mass of the parcel, which is constant in time. Since the circulation loop is confined to a constant density surface, it is also constant, which means that the potential vorticity

$$q = \frac{\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\rho}{\rho} \tag{6.15}$$

is conserved by parcels:

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} q = 0. \tag{6.16}$$

6.2.3 Atmospheric potential vorticity

Similar arguments can be made for the atmosphere, except the upper and lower bounding surfaces are surfaces of constant potential temperature. The *Ertel* potential vorticity in the atmosphere is thus

$$q = \frac{\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\theta}{\rho}. \tag{6.17}$$

In the Boussinesq approximation the density is omitted as it is a constant factor and the potential temperature is replaced by the buoyancy, resulting in the Boussinesq potential vorticity:

$$q = \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla} b \qquad \text{(Boussinesq approximation)}. \tag{6.18}$$

In pressure coordinates the density is $-g^{-1}$, the vorticity is

$$\boldsymbol{\zeta}_a = \left(-\frac{\partial v}{\partial p}, \frac{\partial u}{\partial p}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right), \tag{6.19}$$

and the gradient operator is

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial p}\right), \tag{6.20}$$

so the Ertel potential vorticity becomes

$$q = -g\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\theta \qquad \text{(pressure coordinates)}. \tag{6.21}$$

The terms involving the pressure vertical velocity ω are neglected in equation (6.19) because they are small compared to competing terms when the horizontal scale is much greater than the vertical scale.

Just to see what the Ertel potential vorticity looks like in its full glory, we expand the vorticity and the dot product in equation (6.17), ignoring the horizontal components of the planetary vorticity and the contributions of the vertical velocity to the horizontal vorticity for the same reasons as in pressure coordinates:

$$q = \frac{1}{\rho}\left[-\frac{\partial v}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)\frac{\partial \theta}{\partial z}\right]. \tag{6.22}$$

6.3 Isentropic coordinates

We now consider the atmospheric coordinate system in which the vertical coordinate is potential temperature θ. The main advantage of this coordinate system is that all motions in which there is no heating ($d\theta/dt = S = 0$) are "horizontal", i.e., they take place on constant θ surfaces. As with pressure coordinates, the slope of constant potential temperature surfaces is shallow under most circumstances, allowing a quasi-Cartesian treatment. Above the atmospheric boundary layer, the potential temperature generally increases monotonically with height. Within the daytime convective boundary layer the potential temperature is essentially constant with height (except at very low levels), so in order for isentropic coordinates to be non-singular in the boundary layer, it is necessary to impose an artificial weak increase in potential temperature with height there. The main disadvantage of isentropic coordinates is that the lower boundary is in general not flat, and it changes in response to the fluid flow. It shares this property with pressure coordinates.

The flow velocity on the x-y-θ grid is defined to be

$$\boldsymbol{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{d\theta}{dt}\right) = (u, v, S) = (\boldsymbol{v}_h, S) \tag{6.23}$$

where $\boldsymbol{v}_h = (u, v)$ is the horizontal velocity, i.e., along constant θ surfaces. The density in isentropic coordinates σ is the mass per unit volume in x-y-θ space and is related to the density in geometric coordinates ρ by

$$\sigma d\theta = \rho dz. \tag{6.24}$$

We infer the mass continuity equation in isentropic coordinates by methods we have used previously to be

$$\frac{\partial \sigma}{\partial t} + \boldsymbol{\nabla}_h \cdot (\sigma \boldsymbol{v}_h) + \frac{\partial}{\partial \theta}(\sigma S) = 0 \tag{6.25}$$

where $\boldsymbol{\nabla}_h$ is the gradient along isentropic surfaces. If no heating occurs, the last term on the left side of this equation vanishes.

The "horizontal" momentum equation contains the component of gravity along tilted isentropic surfaces, $-\boldsymbol{\nabla}_h\Phi$, where Φ is the geopotential, as well as the pressure gradient and Coriolis forces. Since the pressure (or Exner function) gradient is along isentropic surfaces, we have $\theta\boldsymbol{\nabla}_h\Pi = \boldsymbol{\nabla}_h(\theta\Pi)$. The horizontal momentum equation thus becomes

$$\frac{d\boldsymbol{v}_h}{dt} + \boldsymbol{\nabla}_h M + f\boldsymbol{k} \times \boldsymbol{v}_h = 0 \tag{6.26}$$

where

$$M = \theta\Pi + \Phi \tag{6.27}$$

is the *Montgomery potential.* The total time derivative has its usual meaning translated to isentropic coordinates:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h + S\frac{\partial}{\partial\theta}. \tag{6.28}$$

Using equation (6.24), the hydrostatic equation becomes

$$\frac{\partial p}{\partial\theta} = -g\sigma. \tag{6.29}$$

In the atmosphere the hydrostatic equation may be written in the alternative form $\theta d\Pi = -gdz = -d\Phi$. Combining this with the differential of the Montgomery potential $dM = \Pi d\theta + \theta d\Pi + d\Phi$ results in the condition

$$\frac{\partial M}{\partial\theta} = \Pi. \tag{6.30}$$

Using this in conjunction with equation (6.27) results in a diagnostic equation for the geopotential:

$$\Phi = M - \theta\frac{\partial M}{\partial\theta}. \tag{6.31}$$

Finally, combining equations (6.29) and (6.30) yields a direct relationship between M and σ:

$$\frac{\partial^2 M}{\partial\theta^2} = -\frac{g}{\rho\theta}\sigma. \tag{6.32}$$

We have used $\Pi = C_p(p/p_R)^\kappa = C_pT/\theta$, $R = \kappa C_p$, and the ideal gas law in deriving this expression.

One additional equation and an upper boundary condition are needed to complete the set of isentropic equations. The surface potential temperature θ_B evolves according to the equation

$$\frac{\partial\theta_B}{\partial t} + \boldsymbol{v}_B \cdot \boldsymbol{\nabla}\theta_B = S_B \tag{6.33}$$

where a subscripted B indicates a value on the lower boundary. If the upper boundary is high in the atmosphere, it is generally sufficient to specify a constant pressure value p_T there. Given this, equation (6.29) can be integrated down to the surface, thus obtaining a vertical profile of pressure at each point in the x-y plane. Equation (6.30) can then be integrated upward from the surface to obtain a profile of Montgomery potential. The momentum (6.26), mass continuity (6.25), and surface potential temperature (6.33) equations can then be stepped forward in time. The cycle is then repeated.

A similar treatment exists for the ocean in which the density is used as the vertical coordinate. These are called *isopycnal* coordinates. Both isentropic and isopycnal coordinates bear a strong resemblance to the shallow water equations extended to multiple layers of decreasing density with height.

The Ertel potential vorticity takes a particularly simple form in isentropic coordinates:

$$q = \frac{\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\theta}{\rho} = \frac{\zeta_a}{\sigma}\left(\frac{\partial\theta}{\partial\theta}\right) = \frac{\zeta_a}{\sigma} = \frac{1}{\sigma}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right), \tag{6.34}$$

where ζ_a is the component of absolute vorticity normal to isentropic surfaces. This is another major advantage of isentropic coordinates.

6.3.1 Anelastic approximation

An approximation to the full isentropic coordinate equations which is analogous to the anelastic approximation is obtained by setting ρ to some ambient profile $\rho_0(\theta)$ in equation (6.32), resulting in

$$\frac{\partial^2 M}{\partial\theta^2} = -\frac{g}{\rho_0\theta}\sigma. \tag{6.35}$$

The advantage of this relatively minor approximation is that the coefficient of σ in the above equation depends only on θ. The mass continuity (6.25) and momentum (6.26) equations remain unchanged, as does the definition of potential vorticity (6.34).

Further progress comes from invoking equation (6.24), which tells us that $\rho_0 = \sigma_0\Gamma$ where $\Gamma(\theta) \equiv (dz_0/d\theta)^{-1}$ is the vertical gradient of ambient potential temperature, $z_0(\theta)$ is the ambient profile of height as a function of potential temperature, and where $\sigma_0 = \rho_0(dz_0/d\theta)$ is the vertical profile of ambient density in isentropic coordinates. Recognizing that $(g/\theta)\Gamma = N^2(\theta)$ is the square of the Brunt-Väisälä frequency, equation (6.35) becomes

$$\frac{\partial^2 M}{\partial\theta^2} = -\frac{N^2}{\sigma_0\Gamma^2}\sigma \qquad \text{(anelastic)}. \tag{6.36}$$

Since N and Γ are familiar meteorological variables, this is a useful form of the M-σ relation.

6.3.2 Boussinesq approximation

An additional approximation analogous to the Boussinesq approximation in geometric coordinates is to replace Γ and N by constant reference values Γ_R and $N_R = (g/\theta_R)\Gamma_R$ in equation

(6.36),

$$\frac{\partial^2 M}{\partial \theta^2} = -\frac{N_R^2}{\sigma_0 \Gamma_R^2}\sigma \qquad \text{(Boussinesq)}, \tag{6.37}$$

where θ_R is a constant reference value of θ. We also assume that σ_0 is constant. These approximations are equivalent to those made in the geometric coordinate Boussinesq case.

One additional approximation is made. The geopotential diagnostic equation (6.31) is replaced by the simpler equation

$$\Phi = M_R - \theta_R \frac{\partial M}{\partial \theta} \tag{6.38}$$

where $M_R = \theta_R C_p$ is a constant reference value of the Montgomery potential. This comes from a scale analysis, which assumes that the range $\Delta\theta$ over which potential temperature varies in the Boussinesq approximation is much less than θ_R. Thus, variations in Φ result much more from variations in $\theta(\partial M/\partial\theta) \approx \theta_R(\Delta M/\Delta\theta)$ than from individual variations in Montgomery potential or potential temperature, allowing these variables to be approximated by constant reference values M_R and θ_R outside of the derivative. From equation (6.30), we see that a simplified relationship between geopotential and Exner function follows:

$$\Phi = \theta_R(C_p - \Pi). \tag{6.39}$$

As in geometric coordinates, the isentropic coordinate Boussinesq approximation is technically valid for disturbances with a vertical scale much less than the scale height of the atmosphere. Its use over deeper layers is purely qualitative, with the purpose of facilitating simple solutions which have similar behavior to the exact equations, but which are computationally less demanding.

6.4 References

Haynes, P. H., and M. E. McIntyre, 1987: On the evolution of vorticity and potential vorticity in the presence of diabatic heating and frictional or other forces. *J. Atmos. Sci.*, **44**, 828-841.

Haynes, P. H., and M. E. McIntyre, 1990: On the conservation and impermeability theorems for potential vorticity. *J. Atmos. Sci.*, **47**, 2021-2031. This and the previous paper introduce the "impermeability theorem" for potential vorticity. Both papers are extremely insightful and well worth absorbing.

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics.* Cambridge University Press, 745 pp. Chapter 3 introduces isentropic coordinates and notes the analogy of these coordinates with shallow water flow. Potential vorticity is introduced in chapter 4.

6.5 Questions and problems

1. Using conservation of circulation for a horizontal circulation loop (shallow water case) or a loop embedded in an isentropic surface (isentropic coordinates), answer the following questions. "Relative vorticity" means the component of relative vorticity normal to the surface bounded by the loop in this case.

   (a) If the relative vorticity is initially zero, how does it change of the area of the circulation loop decreases? The parcel in question is at some fixed positive latitude.

   (b) If the circulation is zero around a parcel at the equator, how does the relative vorticity change as the parcel moves to some positive latitude without changing area?

   (c) In the shallow water case, if a circulation loop starting at a large positive latitude moves toward the equator while maintaining zero relative vorticity, how does the loop area change, and hence the layer thickness?

2. Postulate a substance analogous to a chemical substance, such as the mass of oxygen in a container of air, called the *potential vorticity substance*. The potential vorticity substance is given by

$$Q = \int \rho q dV$$

   where the density ρ and the volume element dV are defined to be consistent with the coordinate system being used. In this picture, ρq is the density of potential vorticity substance and q is the associated mixing ratio. Prove that the amount of potential vorticity substance between two isentropic surfaces never changes. You may wish to read the papers by Haynes and McIntyre (1987, 1990).

3. Derive governing equations for isopycnal coordinates in the ocean in analogy with the atmospheric isentropic equations.

4. Basic modes of the Boussinesq isentropic equations:

   (a) Linearize the Boussinesq form of the isentropic coordinate equations about a state of rest with no rotation or heating, assuming that $M = M_0(\theta) + M'$ and $\sigma = \sigma_0 + \sigma'$ where σ_0 is constant.

   (b) Estimate the value of σ_0 in the troposphere given that $p_s = 10^5$ Pa and $\theta_s = 300$ K at the surface while $p_t = 10^4$ Pa and $\theta_t = 350$ K at the tropopause. Hint: The mass per unit area between the surface and the tropopause is $(p_s - p_t)/g$.

   (c) Assume a plane wave of the form $\exp\left[i\left(kx + m\theta - \omega t\right)\right]$ in the linearized Boussinesq isentropic equations and obtain the dispersion relation for the fundamental modes of this system.