Chapter 12

Equatorial Dynamics

Though heating and friction are important throughout the atmosphere, these phenomena are coupled much more intimately to the dynamics in the tropics and equatorial regions than in middle latitudes. The Coriolis force is also weak compared to middle latitudes, which has important consequences. For these reasons, our understanding of atmospheric dynamics at low latitudes is less advanced than for mid-latitude dynamics.

We begin with a discussion of the factors which distinguish the tropics and equatorial regions from middle latitudes. We then say just a few words about the tropical Hadley and related circulations. Finally, we investigate fundamental linearized modes in the equatorial region. The coupling of convection to these modes is then discussed. We finally say a few words about the Madden-Julian oscillation, which is the most important source of rainfall variance in the equatorial region.

12.1 Weak temperature gradient approximation

Charney (1963) showed using a scale analysis that the small value of the Coriolis parameter in the equatorial regions tends to reduce the coupling between between different levels in the atmosphere. To illustrate this in another way, imagine a quasi-geostrophic potential vorticity anomaly on an f-plane of the form $q^*/q_0 = \epsilon\delta(\theta - \theta_0)\sin(kx)$. The Montgomery potential anomaly generated by this potential vorticity anomaly, assumed also to be proportional to $\sin(kx)$, comes from solving the inversion equation

$$\frac{\partial^2 M^*}{\partial \theta^2} - m^2 M^* = \frac{\epsilon N_R^2}{\Gamma_R^2}\delta(\theta - \theta_0)\sin(kx) \tag{12.1}$$

where

$$m = \frac{kN_R}{f_0 \Gamma_R}. \tag{12.2}$$

The solution is

$$M^* = -\frac{\epsilon N_R^2}{2m\Gamma_R^2}\exp(-m|\theta-\theta_0|)\sin(kx), \tag{12.3}$$

which shows that M^* decreases exponentially away from the $\theta = \theta_0$ level with a decay length of m^{-1}. For a given horizontal scale k^{-1}, the potential vorticity anomaly therefore has a *penetration depth* of at most a few times

$$\delta\theta_p = \frac{1}{m} = \frac{f_0\Gamma_R}{kN_R}. \tag{12.4}$$

Since this is proportional to f_0, the penetration depth gets smaller as one approaches the equator. As Charney points out, different levels in the equatorial atmosphere therefore communicate primarily via the effects of deep convection, which can take near-surface parcels to the tropopause. This greatly reduces the effectiveness of baroclinic instability in the tropics, which depends on the interaction between vorticity anomalies at different levels.

A related result is that horizontal gradients of temperature are weak in the tropics. The Boussinesq thermal wind equation in geometric coordinates is

$$\boldsymbol{\nabla}_h b = -f\boldsymbol{k}\times\frac{\partial \boldsymbol{v}_g}{\partial z}, \tag{12.5}$$

which shows that for a given vertical shear of the geostrophic wind, the associated horizontal buoyancy (or potential temperature) gradient decreases as f decreases. Sobel et al. (2001) developed a simplified model of tropical dynamics based on the idea that horizontal temperature gradients are weak in the tropics. In the limit in which the gradients are zero and the mean profile $\theta = \theta_0(z)$ does not change with time, the thermodynamic equation in geometric coordinates

$$\frac{\partial\theta}{\partial t} + \boldsymbol{v}_h\cdot\boldsymbol{\nabla}_h\theta + w\frac{\partial\theta}{\partial z} = S \tag{12.6}$$

reduces to

$$w\frac{d\theta_0}{dz} = S. \tag{12.7}$$

In this limit there is a one-to-one relationship between the vertical velocity w and the heating S due to convection and radiation. This relationship tends to break down in the lowest part of the tropical atmosphere (say, the lowest kilometer) as a result of spatial variations in the sea surface temperature and in land conditions. Other than that, it is observed to hold within $1-2$ K tropics-wide. In the absence of heating, motions in the tropics are almost purely horizontal. Radiative cooling causes a broad-scale, gentle subsidence, while ascent is limited to the strong but isolated updrafts associated with deep convection.

In parallel with the development of the weak temperature gradient model of tropical dynamics, Sobel et al. (2001) proposes a simple model for convection and the associated precipitation. The basic idea is that the precipitation rate (at least over warm tropical oceans) is a steep function of the column relative humidity or saturation fraction, which is the vertical integral of water vapor density divided by the vertical integral of saturated water vapor density. Bretherton et al. (2004) provides strong observational evidence supporting this hypothesis. Normally, convection reduces the saturation fraction, which in turn reduces the intensity of subsequent convection and precipitation. However, if the surface flux of moisture is sufficiently strong where there is convection, or if the cloudiness associated with convection reduces the radiative cooling and subsidence due to its opacity to infrared radiation, the net effect may be an increase in saturation fraction in response to convection. In this case, an instability exists in which convection and precipitation increase with time, at least until nonlinear or other effects intervene. Such an instability is called *moisture mode instability*. See Raymond et al. (2009) for a review of this subject.

12.2 Hadley and monsoon circulations

The tropics and subtropics are dominated by a meridional circulation called the Hadley circulation. Air rises in convection in the equatorial or near-equatorial regions and subsides radiatively in the subtropics. The monsoon circulation is similar except that rising air occurs a significant distance off the equator in the summer hemisphere, usually with the strongest ascent over the subtropical continents. The corresponding subsidence occurs mostly in the subtropical regions of the winter hemisphere. Since both of these flows can be considered to be zonally symmetric to zeroth order, conservation of angular momentum plays an important role in determining the zonal wind field. On an equatorial beta-plane this conservation property may be expressed by the zonally symmetric zonal momentum equation

$$\frac{d(u - \beta y^2/2)}{dt} = \frac{dm}{dt} = F_x \tag{12.8}$$

where

$$m = u - \beta y^2/2 \tag{12.9}$$

is related to the specific angular momentum of air parcels moving about the center of the earth.

To see this, note that the actual angular momentum of an air parcel of unit mass and zonal velocity u is $L = ru + r^2\Omega$ where r is the distance of the parcel from the axis of rotation of the earth and Ω is the earth's angular rotation rate. The quantity r is related to the earth's radius a and the latitude ϕ by $r = a\cos\phi$, so

$$L = ru + a^2\Omega\cos^2\phi = au\cos\phi + a^2\Omega - a^2\Omega\sin^2\phi. \tag{12.10}$$

We now recall that the beta plane approximation ignores the curvature of the earth's surface except for its effect on the latitudinal variability of the Coriolis parameter. In this case the $\cos\phi$ multiplying u is approximated by 1. We also make the small angle approximation on the latitude in the Ω term and note that $y = \phi a$. Finally with the definition of the equatorial value of beta

$$\beta = \left(\frac{df}{dy}\right)_{\phi=0} = 2\Omega\left(\frac{d\sin\phi}{d\phi}\right)_{\phi=0}\frac{d\phi}{dy} = \frac{2\Omega}{a}, \tag{12.11}$$

we find

$$L \approx a\left(u - \beta y^2/2\right) + a^2\Omega = am + a^2\Omega, \tag{12.12}$$

which shows that the actual angular momentum in the equatorial beta plane approximation is linearly related to m. We will refer to m henceforth as "the angular momentum".

One may ask why the Hadley circulation exists at all. A steady, zonally symmetric solution with no meridional circulation exists for the earth's atmosphere, in which the zonal shear is in thermal wind balance with the meridional temperature gradient resulting from local radiative-convective equilibrium at each latitude. In this solution the zonal wind is assumed to be zero at the surface; surface friction would otherwise cause the solution to be non-steady. Because of the westerly shear, the wind is therefore westerly (i.e., toward the east) at all elevations. Thus, near the equator m takes on positive values. This is called *super-rotation.*

If the earth's atmosphere started out at rest, the angular momentum would be zero or negative everywhere, with zero values occurring only on the equator. Since m is conserved for a zonally symmetric flow in the absence of friction, there is no way that the above state of purely zonal flow in thermal wind balance could establish itself. Furthermore, if this idealized pattern were established by some means, even very weak friction in the interior of the flow would ultimately destroy it. Thus, a steady zonal flow at all latitudes in thermal wind balance cannot be realized in nature. The inevitable consequence is that some sort of meridional circulation must exist in the real atmosphere, at least near the equator. This is the origin of the Hadley and monsoon circulations. The book by Lindzen (1990) is a good reference for understanding the Hadley circulation more fully. Vallis (2006) also discusses the Hadley circulation.

12.3 Matsuno modes

The shallow water equations are more applicable to the dynamics of the atmosphere and the ocean in the Tropics than they are in middle latitudes. For this reason we look for small perturbations about a state of rest of the shallow water equations on an equatorial beta plane. This is a problem first analyzed (correctly) by Matsuno (1966) as a part of his doctoral dissertation. The linearized shallow water equations in this case are

$$\frac{\partial u}{\partial t} - \beta y v + c^2 \frac{\partial \eta}{\partial x} = 0 \tag{12.13}$$

$$\frac{\partial v}{\partial t} + \beta y u + c^2 \frac{\partial \eta}{\partial y} = 0 \tag{12.14}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{12.15}$$

We non-dimensionalize these equations by measuring x and y in units of L_R and t in units of T, where we must have $L_R = (c/\beta)^{1/2}$ and $T = (\beta c)^{1/2}$ in order that $\beta \rightarrow 1$ and $c \rightarrow 1$. Assuming a plane wave in the x direction of the form $\exp[i(kx - \omega t)]$ then results in

$$-i\omega u - yv + ik\eta = 0 \tag{12.16}$$

$$-i\omega v + yu + \frac{\partial \eta}{\partial y} = 0 \tag{12.17}$$

$$-i\omega\eta + iku + \frac{\partial v}{\partial y} = 0. \tag{12.18}$$

The quantity L_R is called the *equatorial Rossby radius.*

12.3.1 Kelvin waves

There are two classes of solutions to equations (12.16) and (12.18), those with $v = 0$ and those with $v \neq 0$. The first class has a single member, equatorial Kelvin waves. Equations (12.16) and (12.18) are purely algebraic for $v = 0$ and take the form

$$-i\omega u + ik\eta = 0 \tag{12.19}$$

$$-i\omega\eta + iku = 0. \tag{12.20}$$

The resulting dispersion relation is

$$\omega = \pm k \tag{12.21}$$

and the polarization relation is

$$u = \pm\eta. \tag{12.22}$$

Equation (12.17) becomes

$$\frac{\partial \eta}{\partial y} \pm y\eta = 0 \tag{12.23}$$

with the resulting solution

$$\eta = \eta_0 \exp\left(\mp y^2/2\right) \tag{12.24}$$

where η_0 is a constant. The plus sign causes η to blow up as $y \rightarrow \pm\infty$. Thus, only the solution with $\omega = k$ and $u = \eta$, which has a Gaussian form about the equator, is physically

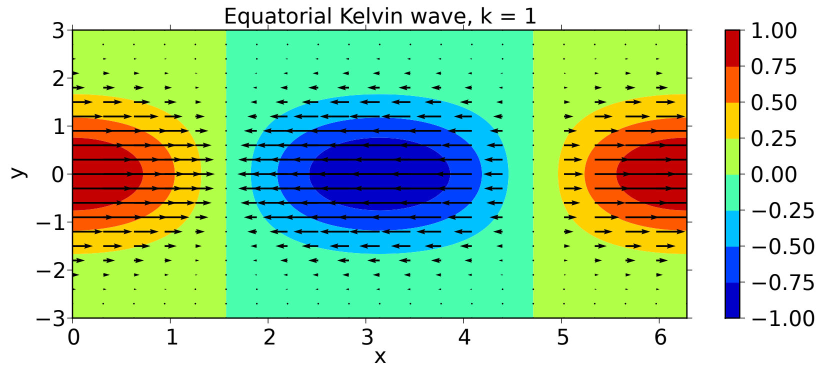

Figure 12.1: Fractional thickness perturbation (color contours) and wind (arrows) for an equatorial Kelvin wave with dimensionless wavenumber $k = 1$.

acceptable. Equatorial Kelvin waves, along with boundary Kelvin waves on an f-plane, are thus "one way" waves. Kelvin waves move only to the east along the equator and the propagation speed is that of non-rotating gravity waves, or c in dimensional terms.

The functional form of Kelvin waves is as follows:

$$\begin{aligned} u =& \eta_0 \exp(-y^2/2) \cos(kx - \omega t) \\ v =& 0 \\ \eta =& \eta_0 \exp(-y^2/2) \cos(kx - \omega t). \end{aligned} \tag{12.25}$$

This is illustrated in figure 12.1 for a wave with $k = 1$.

12.3.2 Equatorial Rossby and gravity waves

Equations (12.16) and (12.18) can be solved for u and η in terms of v:

$$u = \frac{-i\omega yv + ik(\partial v/\partial y)}{k^2 - \omega^2} \qquad \eta = \frac{i\omega(\partial v/\partial y) - ikyv}{k^2 - \omega^2}. \tag{12.26}$$

Substitution of these results into equation (12.17) yields a second order differential equation for v in terms of y:

$$\frac{\partial^2 v}{\partial y^2} + \left(\omega^2 - k^2 - k/\omega - y^2\right) v = 0. \tag{12.27}$$

This equation is of the same form as the Schrödinger equation for the harmonic oscillator and has solutions which go to zero for $y \to \pm\infty$ only if

$$\omega^2 - k^2 - k/\omega = 2n + 1, \qquad n = 0, 1, 2, \ldots. \tag{12.28}$$

This constitutes the dispersion relation for this class of equatorial modes, which we rewrite as

$$\omega^3 - (k^2 + 2n + 1)\omega - k = 0. \tag{12.29}$$

The solutions take the form

$$v = v_0 \exp(-y^2/2) H_n(y) \tag{12.30}$$

where $H_n(y)$ are the Hermite polynomials. The first three Hermite polynomials are

$$\begin{aligned} H_0(y) &= 1 \\ H_1(y) &= 2y \\ H_2(y) &= 4y^2 - 2. \end{aligned} \tag{12.31}$$

In this section we consider only cases with $n > 0$. The $n = 0$ case has some peculiarities, which are considered separate in the next section.

The dispersion relation given by equation (12.29) is cubic and therefore has a very complex exact solution. However, two approximate solutions valid for $k^2 \gg 1$ extract the physics of the situation. If ω^2 is comparable in magnitude to k^2, then we can write equation approximately as

$$\omega^3 - (k^2 + 2n + 1)\omega = 0 \tag{12.32}$$

since $k \ll k^2\omega$ in this case. Discounting the $\omega = 0$ solution, we find

$$\omega \approx \left(k^2 + 2n + 1\right)^{1/2} \tag{12.33}$$

where we have ignored the negative ω solution as well. We can do this with no loss of generality in the case of shallow water solutions because, unlike in quantum mechanics, only the real part of the complex exponential $\exp[i(kx - \omega t)] \to \cos(kx - \omega t)$ has a physical meaning. Thus, as long as we consider both positive and negative values of the wavenumber k, the entire physical domain of $k - \omega$ space is covered, since for every solution $\omega(k)$, there is a corresponding solution $-\omega(-k)$ which is physically indistinguishable from the original. The solutions represented by equation (12.33) are actually inertia-gravity waves, which can move either to the east or to the west.

The other possible solution to equation (12.32) occurs with $|\omega| \ll |k|$. In this case the ω^3 term can be dropped, and the approximate dispersion relation is

$$\omega \approx -\frac{k}{k^2 + 2n + 1}. \tag{12.34}$$

This corresponds to westward-moving equatorial Rossby waves. The functional form of the equatorial Rossby wave with $n = 1$ is

$$\begin{aligned} u =& \eta_0 \left[\frac{\omega y^2 - k(1 - y^2)}{k^2 - \omega^2}\right] \exp(-y^2/2) \sin(kx - \omega t) \\ v =& \eta_0 y \exp(-y^2/2) \cos(kx - \omega t) \\ \eta =& \eta_0 \left[\frac{k y^2 - \omega(1 - y^2)}{k^2 - \omega^2}\right] \exp(-y^2/2) \sin(kx - \omega t) \end{aligned} \tag{12.35}$$

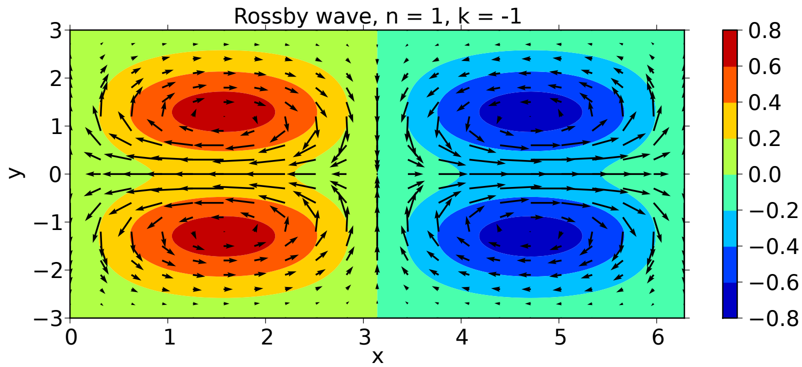

Figure 12.2: As in figure 12.1 except for the equatorial Rossby mode with $n = 1$ and $k = -1$.

and is illustrated for $k = -1$ in figure 12.2.

12.3.3 Mixed Rossby-gravity wave

For the $n = 0$ case the dispersion relation is

$$\omega^3 - (k^2 + 1)\omega - k = 0. \tag{12.36}$$

This factors as follows,

$$(\omega + k)(\omega^2 - k\omega - 1) = 0, \tag{12.37}$$

which produces solutions

$$\omega = -k \tag{12.38}$$

and

$$\omega = \frac{k}{2} \pm \left[\left(\frac{k}{2}\right)^2 + 1\right]^{1/2}. \tag{12.39}$$

The first solution is spurious for the following reason. Substitution of $\omega = -k$ into equation (12.26) shows that this mode has $u = -\eta$. Substituting $\omega = -k$ and $u = -\eta$ into equation (12.16) demonstrates that $v = 0$ for this case. However, if $v = 0$, then we are dealing with a Kelvin mode, and we have already shown that the westward-moving Kelvin mode ($\omega = -k$) has unphysical behavior. Therefore, this mode does not exist in nature.

There initially appears to be two solutions associated with the plus and minus signs in equation (12.39). However, the minus solution results in $\omega < 0$. Furthermore, sending $\omega \to -\omega$ and $k \to -k$ changes the minus sign into the plus sign and vice versa. Thus, these two mathematical solutions correspond to the same physical solution for the reasons

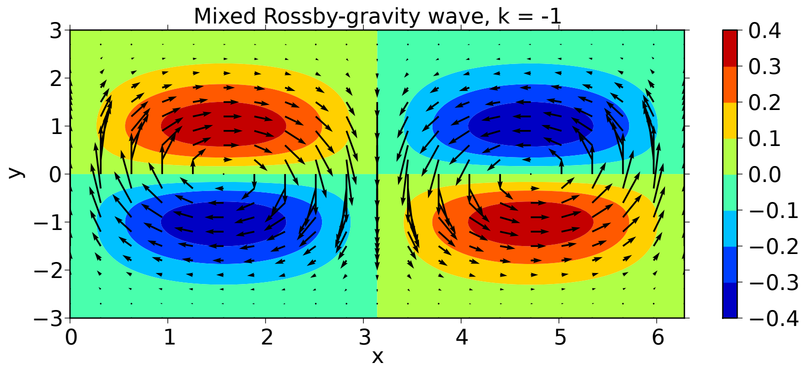

Figure 12.3: As in figure 12.1 except for mixed Rossby-gravity wave with $k = -1$.

expressed in the previous section. Thus, it is sufficient to consider only the case with the plus sign:

$$\omega = \frac{k}{2} + \left[\left(\frac{k}{2}\right)^2 + 1\right]^{1/2}. \tag{12.40}$$

This is the dispersion relation for the mixed Rossby-gravity mode. For phase speeds to the east, the frequency is large and the mode acts like an inertia-gravity wave. Westward-moving modes have small phase speeds and act like Rossby waves. The mixed Rossby-gravity wave has the functional form

$$\begin{aligned} u &= -\eta_0 \omega y \exp(-y^2/2) \sin(kx - \omega t) \\ v &= \eta_0 \exp(-y^2/2) \cos(kx - \omega t) \\ \eta &= -\eta_0 \omega y \exp(-y^2/2) \sin(kx - \omega t). \end{aligned} \tag{12.41}$$

This is illustrated in figure 12.3 for a westward-moving wave with $k = -1$.

Figure 12.4 shows the dispersion relations for the various Matsuno modes. The approximate dispersion relations for the inertia-gravity and Rossby modes given by equations (12.33) and (12.34) are shown for these modes. The other curves are exact.

12.4 Matsuno modes and convection

Wheeler and Kiladis (1999) and others have made spectral analyses of patterns of deep convection in the equatorial regions. They found significant spectral power associated with the dispersion relations (as illustrated in figure 12.4) for Kelvin waves, equatorial Rossby waves, and mixed Rossby-gravity waves. The best fit shallow water equivalent depth for

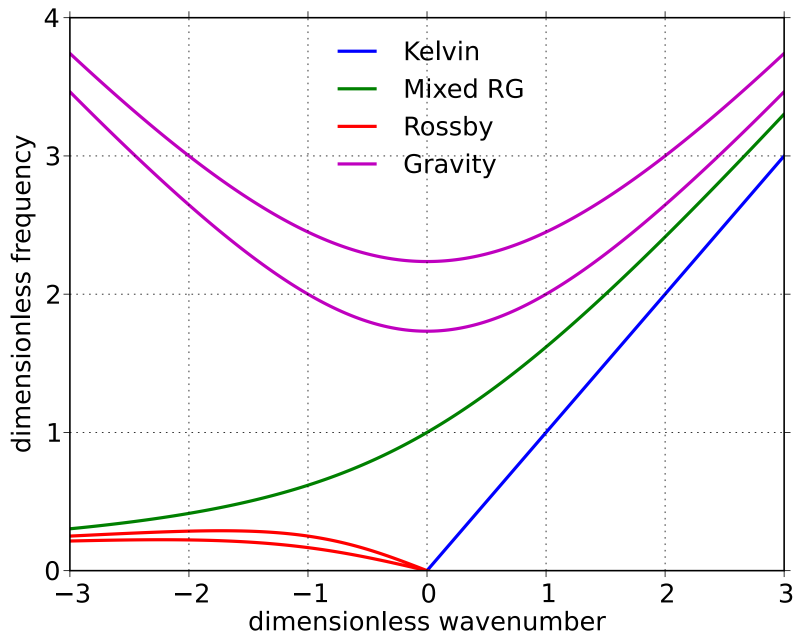

Figure 12.4: Non-dimensionalized dispersion relations for the Matsuno modes. The first two Rossby and gravity modes ($n = 1$ and $n = 2$) are shown. For these modes the approximate dispersion relations (12.33) and (12.34) are used. These differ only slightly from the exact formulas.

these modes is roughly $h_0 = 25$ m. This corresponds to a propagation speed for gravity waves in a non-rotating environment of $c = (gh_0)^{1/2} = 16$ m s^{-1}.

As we showed previously, shallow water modes with a characteristic speed of c are isomorphic to hydrostatic gravity waves with vertical wavenumber

$$m = \frac{N}{c} \qquad (12.42)$$

where N is the Brunt-Väisälä frequency. For $N = 10^{-2}$ s^{-1} and $c = 16$ m s^{-1}, $m = 6.25 \times 10^{-4}$ m^{-1}. However, as a first guess, one might expect that the gravity wave would have a half-vertical wavelength equal to the depth of deep convection, which is essentially the height of the tropopause. In this way, the vertical profile of heating by the convection, which is roughly in the form of a sinusoidal function with a half-wavelength equal to the depth of the convection, would project maximally onto the vertical velocity profile of the gravity wave. In the tropics the tropopause height is about $H = 16$ km, which would imply a vertical wavenumber of $m = \pi/H = 1.96 \times 10^{-4}$ m^{-1}, which is about 1/3 of that implied by observations.

Emanuel et al. (1994) proposed a theory which addressed this problem by hypothesizing that moist convection reduces the effective static stability by virtue of the latent heat release occurring in the updraft. If the moist Brunt-Väisälä frequency N_m is roughly 1/3 of the dry frequency N, then the vertical wavenumber of the convectively coupled waves would become consistent with that observed. The theory proposed by Emanuel et al. (1994) has become known as *convective quasi-equilibrium.*

An alternative theory postulates the existence of a dry Matsuno mode with the observed vertical wavenumber $m = N/c$. This mode is coupled to the convection by a different mechanism than postulated in convective quasi-equilibrium, modulating convection by its effect on the lower tropospheric potential temperature, which controls the degree of convective inhibition. The Matsuno mode feeds off of the heating by the resulting convection, which must occur in the proper phase with the Matsuno mode. This theory is commonly referred to as *moisture-stratiform instability.* The literature is large in this area, but good places to start are Majda et al. (2006), Raymond and Fuchs (2007), and Kuang (2008).

Observations tend to show that moisture-stratiform instability is more in agreement with the vertical structure of the Kelvin mode than is convective quasi-equilibrium. A good summary of observational results is given by Kiladis et al. (2009). Since moisture-stratiform instability depends on the existence of temperature perturbations strong enough to affect convection in the lower troposphere, it operates outside the scope of the weak temperature gradient approximation.

12.5 Madden-Julian oscillation

A great deal of organized tropical convection exists independently of the Matsuno modes, the most obvious being tropical easterly waves, tropical cyclones, and the Madden-Julian oscillation (MJO). The latter shows significant power in the wavenumber-frequency diagram of Wheeler and Kiladis (1999). It is a large-scale, convectively-coupled mode centered in the equatorial regions, but with influence extending into the middle latitudes. The convective part of the MJO typically forms over the Indian Ocean and moves eastward to the central Pacific somewhat irregularly, but at a typical speed of about 5 m s^{-1}. With the exception of some MJO effect on the tropical east Pacific, the convection then disappears. However, a signature of the MJO in 200 hPa winds often propagates completely around the globe, and may be at least partially responsible for initiating the next MJO episode.

A large part of convective variability in the tropics is controlled by the MJO, but it is one of the least-well-understood disturbances in the global atmosphere. A good review article on the MJO is that of Zhang (2005). Raymond and Fuchs (2009) propose that the underlying mechanism of the MJO is the moisture mode instability discussed earlier in this chapter. The problem of the MJO is far from solved and there is a huge current literature on the subject.

12.6 References

Bretherton, C. S., M. E. Peters, and L. E. Back, 2004: Relationships between water vapor path and precipitation over the tropical oceans. *J. Climate*, **17**, 1517-1528.

Charney, J. G., 1963: A note on large-scale motions in the tropics. *J. Atmos. Sci.*, **20**, 607-609.

Emanuel, K. A., J. D. Neelin, and C. S. Bretherton, 1994: On large-scale circulations in convecting atmospheres. *Quart. J. Roy. Meteor. Soc.*, **120**, 1111-1143.

Khouider, B., and A. J. Majda, 2006: A simple multicloud parameterization for convectively coupled tropical waves. Part I: Linear analysis. *J. Atmos. Sci.*, **63**, 1308-1323.

Kiladis, G. N., M. C. Wheeler, P. T. Haertel, K. H. Straub, and P. E. Roundy 2009: Convectively coupled equatorial waves. *Rev. Geophys.*, **47**, RG2003, doi:10.1029/2008RG000266.

Kuang, Z., 2008: A moisture-stratiform instability for convectively coupled waves. *J. Atmos. Sci.*, **65**, 834-854.

Lindzen, R. S., 1990: *Dynamics in Atmospheric Physics.* Cambridge University Press, 310 pp.

Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, **44**, 25-43.

Raymond, D. J., and Ž. Fuchs, 2007: Convectively coupled gravity and moisture modes in a simple atmospheric model. *Tellus*, **59A**, 627-640.

Raymond, D. J., and Ž. Fuchs, 2009: Moisture modes and the Madden-Julian oscillation. *J. Climate*, **22**, 3031-3046.

Raymond, D. J., S. L. Sessions, A. H. Sobel, Ž. Fuchs, 2009: The mechanics of gross moist stability. *J. Adv. Model. Earth Syst.*, **1**, doi:10.3894/JAMES.2009.1.9.

Sobel, A. H., J. Nilsson, and L. M. Polvani, 2001: The weak temperature gradient approximation and balanced tropical moisture waves. *J. Atmos. Sci.*, **58**, 3650-3665.

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics.* Cambridge University Press, 745 pp. The Hadley circulation is discussed in chapter 11.

Wheeler, M., and G. N. Kiladis, 1999: Convectively coupled equatorial waves: Analysis of clouds and temperature in the wavenumber-frequency domain. *J. Atmos. Sci.*, **56**, 374-399.

Zhang, C., 2005: Madden-Julian oscillation. *Rev. Geophys.*, **43**, doi: 0.1029/2004RG000158.

12.7 Questions and problems

1. Compute the penetration depth $\delta\theta_p$ for a disturbance of horizontal wavelength 3000 km at latitudes 5° and 50°. Assume typical tropical atmospheric values $\Gamma_R = d\theta/dz = 5 \times 10^{-3}$ K m^{-1} and $N = 10^{-2}$ s^{-1}.

2. Consider steady, zonally symmetric meridional flow with linear friction $F_x = -\lambda u$ on a beta-plane. The meridional velocity v is assumed to be constant.

   (a) Show that the angular momentum obeys the equation
   $$v\frac{dm}{dy} + \lambda m = -\lambda\beta y^2/2.$$

   (b) Show that if y is measured in units of v/λ and if m is measured in units of $\beta v^2/\lambda^2$, this equation simplifies to
   $$\frac{dm}{dy} + m = -y^2/2.$$

   (c) Solve the above equation, noting that it has homogeneous and inhomogeneous parts to the solution. Hint: Try $m_I = Ay^2 + By + c$ for the inhomogeneous part. (You know what the homogeneous solution is!)

   (d) Use this result to plot the zonal wind in steady, cross-equatorial flow in the boundary layer. Compare in particular the behavior of the zonal wind to that which would occur if there were no friction. As an example, assume an initial $y = -1500$ km, an initial $u = -10$ m s^{-1}, $v = 5$ m s^{-1}, and $\lambda = 10^{-5}$ s^{-1}. At what approximate value of y does the zonal wind switch from easterly to westerly?

3. Compute the group velocities for the Rossby and mixed Rossby-gravity modes. Determine the range of dimensionless wavenumbers for which each has eastward and westward group velocities.

4. For the Kelvin wave compute the relative vorticity and divergence
   $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$
   Compute the ratio $|\Delta|/|\zeta|$ where the absolute value sign indicates the magnitude of the quantity at fixed y and k, obtained by setting the multiplying cosine or sine to unity. Sketch in the $k - y$ plane where this ratio is greater and less than unity.

5. Repeat problem 4 for the case of the mixed Rossby-gravity wave, except plot the divergence-vorticity ratio as a function of k only for $y = 1/2$. Explicitly use the dispersion relation to eliminate ω in favor of k. Pay particular attention to the limits where $k \gg 1$ and $k \ll -1$.