Chapter 11

Heating, Friction, and Ageostrophic Wind

Surface heat fluxes, convection, and radiation act to heat and cool the atmosphere while surface friction extracts momentum from the atmosphere. For time scales exceeding a few hours to a few days, these effects become important for atmospheric motion. Detailed elucidation of these mechanisms demands a complete course of its own, so there is not space here to do this. Instead, we present some of the approximations used to generate highly simplified representations of these processes. We then see how heating and friction play out in geophysical fluid dynamics, especially in their effect on the potential vorticity. The ageostrophic wind plays an important role when heating and friction are added.

11.1 Effects of heating and friction

We first examine the effect of heating and external forces on the potential vorticity. As the analysis is most easily done in isentropic coordinates, we use this coordinate system. We start from the momentum equation with an imposed horizontal specific force $\boldsymbol{F}$ added

$$\frac{d\boldsymbol{v}_h}{dt} + \boldsymbol{\nabla}_h M + f\boldsymbol{k} \times \boldsymbol{v}_h = \boldsymbol{F} \tag{11.1}$$

and the mass continuity equation in advective form

$$\frac{d\sigma}{dt} + \sigma \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{11.2}$$

Note that

$$\boldsymbol{v} = (u, v, S) = (\boldsymbol{v}_h, S) \tag{11.3}$$

where

$$S = \frac{d\theta}{dt} \tag{11.4}$$

and

$$\boldsymbol{\nabla} = \left(\boldsymbol{\nabla}_h, \frac{\partial}{\partial \theta}\right). \tag{11.5}$$

The total time derivative here is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h + S\frac{\partial}{\partial \theta}. \tag{11.6}$$

The heating S in isentropic coordinates enters not as a source term, but as the vertical component of velocity.

The absolute vorticity vector in isentropic coordinates is

$$\boldsymbol{\zeta} = (\boldsymbol{\zeta}_h, \zeta_a) = \left(-\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right). \tag{11.7}$$

Using previously employed methods, the governing equation for the vertical component of absolute vorticity in isentropic coordinates in flux form is

$$\frac{\partial \zeta_a}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{v}\zeta_a - \boldsymbol{\zeta} S + \boldsymbol{k} \times \boldsymbol{F}) = 0. \tag{11.8}$$

The vertical baroclinic generation term vanishes since ζ_a is the component of absolute vorticity normal to isentropic surfaces. The vertical component of the term under the divergence in equation (11.8) is zero, which allows further simplification, but we prefer to keep the equation as it stands, as this facilitates the conversion to advective form:

$$\frac{d\zeta_a}{dt} + \zeta_a \boldsymbol{\nabla} \cdot \boldsymbol{v} - \boldsymbol{\nabla} \cdot (\boldsymbol{\zeta} S - \boldsymbol{k} \times \boldsymbol{F}) = 0 \tag{11.9}$$

As noted previously, the potential vorticity in isentropic coordinates is $q = \zeta_a/\sigma$, which means that

$$\frac{dq}{dt} = \frac{1}{\sigma}\frac{d\zeta_a}{dt} - \frac{\zeta_a}{\sigma^2}\frac{d\sigma}{dt}. \tag{11.10}$$

Eliminating the time derivatives using equations (11.2) and (11.9) yields the advective form of the potential vorticity evolution equation, including the effects of heating and an external force:

$$\frac{dq}{dt} = \frac{1}{\sigma}\left[\boldsymbol{\nabla}_h \cdot (\boldsymbol{\zeta}_h S - \boldsymbol{k} \times \boldsymbol{F}) + \frac{\partial \zeta_a S}{\partial \theta}\right]. \tag{11.11}$$

Writing out the total time derivative and eliminating ζ_a in favor of q yields

$$\frac{\partial q}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h q + S\frac{\partial q}{\partial \theta} = \frac{1}{\sigma}\boldsymbol{\nabla}_h \cdot (\boldsymbol{\zeta}_h S - \boldsymbol{k} \times \boldsymbol{F}) + \frac{1}{\sigma}\frac{\partial \sigma S q}{\partial \theta}. \tag{11.12}$$

Product rule expansion of the last term on the right results in a cancellation with the vertical advection term on the left, producing

$$\frac{\partial q}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h q = \frac{1}{\sigma}\boldsymbol{\nabla}_h \cdot (\boldsymbol{\zeta}_h S - \boldsymbol{k} \times \boldsymbol{F}) + \frac{q}{\sigma}\frac{\partial \sigma S}{\partial \theta}. \tag{11.13}$$

Finally, we replace q by q_g and approximate σ by a constant σ_0 and q_g by a constant q_0 on the right side. We also drop the term $\boldsymbol{\zeta}_h S$ since it is nonlinear in deviations from the base state to produce the Boussinesq, quasi-geostrophic form of this equation:

$$\frac{\partial q_g}{\partial t} + \boldsymbol{v}_g \cdot \boldsymbol{\nabla}_h q_g = -\frac{1}{\sigma_0}\boldsymbol{\nabla}_h \cdot (\boldsymbol{k} \times \boldsymbol{F}) + q_0 \frac{\partial S}{\partial \theta}. \tag{11.14}$$

We retain the ageostrophic wind for reasons discussed above. The first term on the right in equation (11.14) represents the effect of surface or other forms of friction. The second term enhances the potential vorticity when $\partial S/\partial\theta > 0$ and attenuates it when $\partial S/\partial\theta < 0$.

The mass continuity equation is extended slightly in the presence of a heat source, taking the form

$$\frac{\partial \sigma}{\partial t} + \boldsymbol{\nabla}_h \cdot (\sigma \boldsymbol{v}_h) + \frac{\partial \sigma S}{\partial \theta} = 0. \tag{11.15}$$

The last term on the left is associated with the vertical transfer of mass across isentropic surfaces due to the effect of heating. The Boussinesq, quasi-geostrophic form, solved for the Laplacian of the velocity potential is

$$\nabla_h^2 \chi = -\boldsymbol{\nabla}_h \cdot \boldsymbol{v}_a = \frac{1}{\sigma_0}\left(\frac{\partial \sigma}{\partial t} + \boldsymbol{v}_g \cdot \boldsymbol{\nabla}_h \sigma\right) + \frac{\partial S}{\partial \theta}. \tag{11.16}$$

The only change from the adiabatic version is the addition of the last term.

The rest of the apparatus of quasi-geostrophic theory in Boussinesq, isentropic form carries over without change.

11.2 Heating

Solar heat fluxes drive the ocean-atmosphere system. Most of the incoming solar energy is either reflected back to space by clouds or is absorbed by the surface; a small fraction is absorbed by moisture near the surface and ozone in the stratosphere. Generally speaking, the land has low heat capacity and incoming solar energy is returned to the atmosphere on a short time scale, either in terms of sensible or latent (evaporated water substance) heat. In contrast, the oceans have great heat capacity and move absorbed energy around about as efficiently as the atmosphere. Therefore, no global heat budget can be contemplated without detailed consideration of the oceans.

Convection, both dry and moist, act to return solar energy to the atmosphere and to lift it to the upper atmosphere where it is radiated away to space. This infrared radiation to space cools the troposphere by of order $1-2$ K day^{-1}. To zeroth order, convection tends to drive the atmosphere to an adiabatic lapse rate (either dry or moist depending on whether saturation exists) such that rising convective parcels acquire only small positive buoyancy. If other processes act to create a lapse rate more stable than adiabatic, then convection shuts off.

In addition to the upward movement of energy from the surface, there is meridional transport of energy from low to high latitudes. The atmosphere and ocean contribute roughly equally to the poleward transport. As might be expected, this transport is much stronger in the winter hemisphere than in the summer. Combined radiative-convective effects at each latitude along with the surface heat balance drive the atmosphere toward a local radiative-convective equilibrium profile. However, the resulting equilibrium temperatures at high latitudes are very low in winter. The atmospheric and oceanic flows which transport heat toward the poles are largely responsible for moderating this cooling tendency.

Oceanic heat transport occurs to a great extent in basin-scale gyres which carry heat poleward in so-called *western boundary currents* along the east shores of continents. These relatively shallow horizontal gyres are largely driven by the surface stress from atmospheric winds, though there is an additional contribution from deep oceanic circulations. Atmospheric heat transport results from a combination of zonally symmetric overturning flows and poleward transport due to atmospheric eddies associated with baroclinic instability.

A simple way to approximate roughly the large-scale effects of heating and cooling in the atmosphere is via the mechanism of *Newtonian relaxation*, in which the heating rate is given by

$$S = \frac{d\theta}{dt} = \frac{\theta_T - \theta}{\tau} \tag{11.17}$$

where $\theta_T(x, y, z, t)$ is a target distribution of potential temperature, θ is the actual potential temperature, and τ is an appropriately selected time constant. This is of course highly approximate, but it allows us to make idealized calculations.

11.3 Interfacial fluxes

Transfers of heat, moisture, and momentum into or out of the atmosphere or ocean occur across the interface between the fluid in question and the other fluid or the adjacent solid surface. Such transfers are initially confined to a layer of fluid typically thin compared to the dimensions of the fluid body as a whole. This layer is called the *boundary layer*. The boundary layer is generally turbulent, which has the tendency to homogenize the quantities in question in the boundary layer. The flux of an intensive quantity such as potential temperature, water vapor mixing ratio, or specific momentum across a boundary is often represented by a *bulk flux formula*, which encapsulates the effects of the turbulence. For any intensive quantity χ, the bulk flux representation of the boundary flux takes the form

$$T_\chi = \rho_{BL} C U_{eff} (\chi_I - \chi_{BL}) \tag{11.18}$$

where ρ_{BL} is the density of the fluid in the boundary layer, C is an *exchange coefficient* which is dimensionless and typically $1 - 2 \times 10^{-3}$, U_{eff} is the *effective wind* in the boundary layer, to be discussed below, χ_{BL} is the characteristic value of χ in the boundary layer, and χ_I is the value of χ immediately adjacent to the interface.

The effective boundary layer wind is given by

$$U_{eff} = \left(U_{BL}^2 + W^2\right)^{1/2} \tag{11.19}$$

where U_{BL} is the actual wind in the boundary layer and W is a gustiness correction which accounts for the fact that the wind is never completely steady. Boundary layer scaling arguments should tell us how big W should be, but this is beyond our scope. Miller et al. (1992) suggest that $W \approx 3$ m s^{-1} gives reasonable results in global atmospheric models.

For a conserved variable, the boundary layer eddies redistribute associated substance through the depth of the boundary layer. Thus, for instance, a potential temperature flux (really, a heat flux), generates a potential temperature source through a boundary layer depth h equal to

$$S_{BL} = \frac{d\theta_{BL}}{dt} = \frac{T_\theta}{\rho_{BL} h} = \frac{CU_{eff}(\theta_I - \theta_{BL})}{h}, \qquad z < h \tag{11.20}$$

while a surface drag flux would generate a specific force

$$\boldsymbol{F}_{BL} = \frac{CU_{eff}(\boldsymbol{U}_I - \boldsymbol{U}_{BL})}{h}, \qquad z < h. \tag{11.21}$$

For the atmosphere over the ocean, $\boldsymbol{U}_I$ would be the surface ocean flow velocity, often approximated by zero. For the ocean, $\boldsymbol{U}_I$ would be the atmospheric boundary layer wind. Obviously, for an atmosphere over land, we would have $\boldsymbol{U}_I = 0$.

Equations (11.20) and (11.21) are nonlinear, which is problematic in linearized calculations. A simple linearization which is sometimes used in models in which the boundary layer is treated explicitly is to make the approximation

$$\mu = \frac{CU_{eff}}{h} = \text{constant}, \tag{11.22}$$

where suitable values of U_{eff} and h are chosen. An even simpler approximation is to ignore the existence of an explicit boundary layer completely and approximate (for example) equation (11.20) by

$$S_{BL} = \mu \exp(-z/h)(\theta_I - \theta). \tag{11.23}$$

Boundary layers are somewhat problematic with isentropic coordinates, since the potential temperature is nearly constant with height. Thus, the boundary layer is compressed into an infinitely thin layer in isentropic coordinates with a finite amount of mass per unit area, making the isentropic density infinite. A feasible approximation is to compress the dynamics of the boundary layer into its effects on the surface potential temperature and wind speed, such that

$$S_B = \mu(\theta_I - \theta_B). \tag{11.24}$$

This requires an independent estimate of the thickness of the boundary layer h in geometric coordinates as well as the effective wind speed there.

11.4 Moist convection

Moist convection is an irreversible heat source in the atmosphere because the condensed water produced by latent heat release mostly falls out as precipitation. Since condensation occurs in the upward branch of the the convective circulation, this is where the heat is released. Some precipitation is evaporated if it falls through unsaturated air. This generally produces cooling in downdrafts. However, if at least some of the precipitation reaches the surface, then the total heating exceeds the total cooling.

As convection is ultimately a closed circulation, the air lifted and heated in the updrafts ultimately has to return to low levels. However, except for the air in downdrafts produced by evaporative cooling, the return flow sinks gradually over a large area as a result of radiative cooling. Thus, moist convection spans a broad range of scales, with upward motions generally occurring on small scales, while downward motion is a large-scale phenomenon. This makes moist convection particularly hard to deal with in geophysical fluid dynamics.

Moist convection has a large effect on radiative transfer in the atmosphere due to the fact that the cloudy outflow at middle and high levels is generally opaque to infrared radiation and can cover large areas. Clouds also reflect a large fraction of incident solar radiation back to space.

Convection also transports horizontal momentum vertically. This transport depends on the detailed structure of the convection. The vertical divergence of the vertical flux of horizontal momentum acts like a force on the atmosphere.

11.5 Other forms of friction

Flow over terrain can introduce friction into the atmosphere. Pressure tends to be higher on windward compared to leeward slopes, which exerts a retarding stress on the atmosphere. This stress is often propagated upward via gravity waves, which by the non-interaction theorem do not deposit their momentum until they dissipate. This form of friction is very difficult to treat in a quantitative fashion, as it depends in detail on gravity wave dynamics.

Friction also occurs when shear or convective instability is manifested. Simple qualitative treatments of this type of instability are possible using a Richardson number criterion plus scaling arguments. We will not deal with these two sources of friction here.

11.6 References

Haynes, P. H., and M. E. McIntyre, 1987: On the evolution of vorticity and potential vorticity in the presence of diabatic heating and frictional or other forces. *J. Atmos. Sci.*, **44**, 828-841.

Haynes, P. H., and M. E. McIntyre, 1990: On the conservation and impermeability theorems for potential vorticity. *J. Atmos. Sci.*, **47**, 2021-2031. These papers introduce the treatment of heating and friction in the potential vorticity equations.

11.7 Questions and problems

1. For a Boussinesq, non-rotating atmosphere initially at rest, assume heating of the form
$$S = S_0\delta(y)\cos\left[\pi(\theta-\theta_M)/(2\Delta\theta)\right], \qquad -\Delta\theta < \theta < \Delta\theta$$
is applied starting at time $t = 0$, where S_0, θ_M, and $\Delta\theta$ are constants. No potential vorticity is generated in this case, so $q^* = 0$, $M^* = 0$, and $\sigma^* = 0$. (Starred quantities have to do with motion.) Compute the ageostrophic wind directly from the quasi-geostrophic mass continuity equation (11.16). Make a sketch of the heating and the flow pattern.

2. Redo problem 1 in a rotating atmosphere with constant Coriolis parameter via the following steps:
   (a) Compute q^* as a function of time using equation (11.14), ignoring friction and the geostrophic wind.
   (b) From q^* compute the perturbation Montgomery potential M^*.
   (c) From M^* compute $\boldsymbol{v}_g$ and $\sigma*$. Was the assumption of ignoring the geostrophic wind in the calculation of q^* justified?
   (d) From σ^* compute the ageostrophic wind using equation (11.16). How does this result differ from the result of the above problem?

3. Compute the response of the linearized, Boussinesq, hydrostatic primitive equations
$$\frac{\partial v}{\partial t} + \frac{\partial \Pi'}{\partial y} = 0$$
$$\frac{\partial \Pi'}{\partial z} - b' = 0$$
$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\frac{\partial b'}{\partial t} + N^2 w = S$$
of an atmosphere initially at rest to a heating profile with spatial structure
$$S = \frac{db}{dt} = S_0\delta(y)\sin(mz)H(t) \qquad 0 < z < \pi/m$$
where $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t > 0$. Assume constant Brunt-Väisälä frequency N. Solve the problem by taking the following steps:

(a) Assuming that $w, b' \propto \sin(mz)$ and $v, \Pi \propto \cos(mz)$, show that these equations can be used to derive a forced wave equation for v:

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial y^2} = \frac{S_0}{m} \delta'(y) H(t),$$

where $\delta'(y) = d\delta(y)/dy$ and $c^2 = N^2/m^2$.

(b) Show that this equation has a solution of the form

$$v = C \left[H(y)H(ct - y) - H(-y)H(ct + y) \right] \cos(mz).$$

Hint: Note that $dH(y)/dy = \delta(y)$ and use the chain rule where needed.

(c) Make sketches (not computer graphs!) which allow you to interpret this solution. In particular, also show the vertical velocity and the buoyancy perturbation and compare with the solution to problem 1.

4. Frictional spindown. Consider a slab-symmetric f-plane with $\partial/\partial x = 0$, no heating, but with frictional force $F_x = -\lambda u_B \exp[-\mu(\theta - \theta_B)]$, $F_y = 0$, where $u_B(y, t)$ is the surface zonal wind, θ_B is the (constant) surface potential temperature, and λ and μ are constants.

(a) Show in the quasi-geostrophic case that the potential vorticity advection equation (11.14) reduces to

$$\frac{\partial q^*}{\partial t} = -\frac{1}{\sigma_0} \frac{\partial F_x}{\partial y},$$

so that we can write the potential vorticity inversion equation as

$$\frac{1}{f_0^2} \frac{\partial^2 M_t^*}{\partial y^2} + \frac{\Gamma_R^2}{N_R^2} \frac{\partial^2 M_t^*}{\partial \theta^2} = -\frac{1}{f_0} \frac{\partial F_x}{\partial y}$$

where $M_t^* = \partial M^* / \partial t$.

(b) Suppose that $u_B = u_0 \sin(ly)$ at time $t = 0$ and solve for M_t^* at that time, maintaining zero potential temperature perturbation at the surface, which implies that

$$\left(\frac{\partial M_t^*}{\partial \theta} \right)_{\theta_B} = 0.$$

The solution for M_t^* has an inhomogeneous part $\propto \cos(ly) \exp[-\mu(\theta - \theta_B)]$ and a homogeneous part $\propto \cos(ly) \exp[-m(\theta - \theta_B)]$, with m to be determined.

(c) From M_t^* find $\partial \sigma^* / \partial t$ and use equation (11.16) to solve for the ageostrophic wind v_a as a function of y and θ at $t = 0$.

(d) Sketch v_a as a function of y at the surface and compare the spatial dependence to that of the initial (geostrophic) surface wind u_B. Examine the vertical structure of v_a and its magnitude as a function of the meridional wavenumber l of the geostrophic wind. In particular, compute

$$\int_{\theta_B}^{\infty} v_a d\theta.$$

(e) Compare the magnitude of v_a at the surface with the surface meridional wind predicted by the zonal momentum equation (shown below) in steady state and zonal symmetry:

$$\frac{\partial u}{\partial t} + \frac{\partial M}{\partial x} - fv = F_x.$$