Chapter 2

Convection and turbulence

2.1 Boussinesq energetics

Here we recap the governing equations in the Boussinesq approximation for later use and show how energy is transported and transformed.

The momentum equation in the Boussinesq approximation takes the form

$$\frac{d\mathbf{v}}{dt} + \nabla\pi - b\hat{\mathbf{z}} + 2\Omega \times \mathbf{v} = 2\nu\nabla \cdot \mathbf{D} \tag{2.1}$$

where we have retained the strain rate form of the viscous term:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{2.2}$$

Mass continuity is given by

$$\nabla \cdot \mathbf{v} = 0. \tag{2.3}$$

We now arbitrarily add a diffusion term to the buoyancy equation, based on the idea that the diffusive flux of buoyancy is given by $-\kappa\nabla b$ where κ is the kinematic diffusivity, so that the diffusive source of buoyancy is $\nabla \cdot (\kappa\nabla b)$. We assume that the diffusivity is constant, so that the governing equation for buoyancy becomes

$$\frac{db}{dt} = \kappa\nabla^2 b. \tag{2.4}$$

It is important to note that this formulation hides a variety of sins! For instance, in the case of an incompressible fluid, the addition of diffusion means that equation (2.3) should have a diffusion term as well, since the full mass continuity equation does not. We resolutely ignore this small error. For an ideal gas, where the buoyancy equation is derived from the entropy equation, this formulation ignores the irreversible generation of entropy, which can be important under certain circumstances. This issue will be discussed in greater detail later. In spite of the above issues, the Boussinesq equations retain a special interest for approximate studies of atmospheric flows due to their simplicity.

We now demonstrate the energy conservation properties of the Boussinesq equations. Dotting equation (2.1) with the velocity results in

$$\frac{dv^2/2}{dt} + \mathbf{v} \cdot \nabla \pi - b v_z = 2\nu \mathbf{v} \cdot (\nabla \cdot \mathbf{D}). \tag{2.5}$$

The Coriolis term $\mathbf{v} \cdot (\Omega \times \mathbf{v}) = 0$ and hence does not appear in this equation. The quantity $v^2/2$ is the kinetic energy per unit mass. However, given the assumption in the Boussinesq approximation that the mass density is constant except in the buoyancy term, it is also proportional to the kinetic energy per unit volume. The second term in the above equation can be written $\nabla \cdot (\mathbf{v}\pi)$ by virtue of the mass continuity equation (2.3). Furthermore, we can expand the parcel derivative of $v^2/2$

$$\frac{dv^2/2}{dt} = \frac{\partial v^2/2}{\partial t} + \mathbf{v} \cdot \nabla (v^2/2) = \frac{\partial v^2/2}{\partial t} + \nabla \cdot (\mathbf{v} v^2/2) \tag{2.6}$$

by the same reasoning. Finally,

$$b v_z = b \frac{dz}{dt} = \frac{dbz}{dt} - z \frac{db}{dt} = \frac{\partial bz}{\partial t} + \nabla \cdot (\mathbf{v} bz) - \kappa z \nabla^2 b. \tag{2.7}$$

Substituting these relations into equation (2.5) results in

$$\frac{\partial}{\partial t}\left(v^2/2 - bz\right) + \nabla \cdot \left[\mathbf{v}\left(v^2/2 - bz + \pi\right)\right] = -\kappa z \nabla^2 b + 2\nu \mathbf{v} \cdot (\nabla \cdot \mathbf{D}), \tag{2.8}$$

where the terms involving viscous and diffusive processes have been placed on the right side of the equation.

The viscous and diffusive terms can be manipulated to further advantage:

$$z \nabla^2 b = \nabla \cdot (z \nabla b) - \nabla z \cdot \nabla b = \nabla \cdot (z \nabla b - b \hat{\mathbf{z}}) \tag{2.9}$$

and

$$\mathbf{v} \cdot (\nabla \cdot \mathbf{D}) = v_j \frac{\partial D_{ij}}{\partial x_i} = \frac{\partial v_j D_{ij}}{\partial x_i} - \frac{\partial v_j}{\partial x_i} D_{ij} = \nabla \cdot (\mathbf{v} \cdot \mathbf{D}) - D_{ji} D_{ij} \tag{2.10}$$

where we have used the fact that $\partial v_j/\partial x_i = R_{ji} + D_{ji}$ and that $R_{ji}D_{ij} = 0$ by symmetry considerations. Substituting these into equation (2.8) results in

$$\frac{\partial}{\partial t}\left(v^2/2 - bz\right) + \nabla\cdot\left[\mathbf{v}\left(v^2/2 - bz + \pi\right) + \kappa(z\nabla b - b\hat{\mathbf{z}}) - 2\nu\mathbf{v}\cdot\mathbf{D}\right] = -2\nu|\mathbf{D}|^2 \tag{2.11}$$

where $|\mathbf{D}|^2 = D_{ji}D_{ij}$ is the sum of the squares of the components of the strain rate tensor, and therefore is positive definite.

This equation is in the form

$$\frac{\partial e}{\partial t} + \nabla\cdot\mathbf{F}_e = S_e \tag{2.12}$$

where e is the total mechanical energy per unit volume (assuming that the density equals unity) with $v^2/2$ being the kinetic energy and $-bz$ the gravitational potential energy. The latter is larger when the buoyancy is smaller at higher levels, i. e., the density is greater there – this explains the minus sign.

The quantity $\mathbf{F}_e$ is the energy flux. The part $\mathbf{v}(v^2/2 - bz)$ is the flux of energy due to mass transport while $\mathbf{v}\pi$ represents energy transport due to pressure work of one part of the fluid on another. The balance is the flux due to viscosity and to diffusion of buoyancy.

On the right side of equation (2.12) is the energy source term $S_e = -2\nu|\mathbf{D}|^2$. Since this is never positive, the source is actually an energy sink due to the action of viscosity in converting mechanical energy into internal energy, i. e., heat. This is the only interchange in the Boussinesq system between mechanical and internal energy. Thus, the Boussinesq approximation does not include the fluid dynamics of heat engines.

2.2 Boussinesq stability analysis

We begin by analyzing the stability of a horizontally homogeneous fluid at rest. We use the Boussinesq approximation with viscosity and diffusion ignored for simplicity, and assume an ambient buoyancy profile of the form $b_0(z) = \gamma z$, where γ is a constant. Positive γ corresponds to the potential temperature increasing with height in an ideal gas, or the density decreasing with height in an incompressible fluid. We further define an ambient kinematic pressure profile $\pi_0(z)$ in hydrostatic balance with b_0:

$$\frac{d\pi_0}{dz} = b_0. \tag{2.13}$$

Writing the buoyancy and kinematic pressure as the sum of the ambient profiles plus a small perturbation, $b = b_0 + b'$, $\pi = \pi_0 + \pi'$, and assuming the velocity is the same order of smallness as b' and π', we linearize the Boussinesq equations in terms of small quantities:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \pi' - b'\hat{\mathbf{z}} = 0, \tag{2.14}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2.15}$$

$$\frac{\partial b'}{\partial t} + \gamma v_z = 0. \tag{2.16}$$

We now try plane wave solutions of the form $(\mathbf{v}, \pi', b') \propto \exp[i(k_x x + k_z z - \omega t)]$, where the omission of the y component of the wave vector does not represent a loss of generality, due to the symmetry of the governing equations to rotations about a vertical axis. The resulting equations can be written in matrix form:

$$\begin{pmatrix} -i\omega & 0 & ik_x & 0 \\ 0 & -i\omega & ik_z & -1 \\ ik_x & ik_z & 0 & 0 \\ 0 & \gamma & 0 & -i\omega \end{pmatrix} \begin{pmatrix} v_x \\ v_z \\ \pi' \\ b' \end{pmatrix} = 0, \tag{2.17}$$

with the additional condition $-i\omega v_y = 0$, or $v_y = 0$ for $\omega \neq 0$. This equation has non-trivial solutions only when the determinant of the coefficients of the square matrix is zero, which leads us to the following dispersion relation:

$$\omega^2 = \frac{k_x^2 \gamma}{k_x^2 + k_z^2}. \tag{2.18}$$

Let us first consider the case $\gamma > 0$ and set $\gamma = N^2$. The quantity N is called the Brunt-Väisälä frequency. In this case the equilibrium state of rest is stable, in that small perturbations of sinusoidal form do not grow with time, but oscillate, due to the fact that ω is real. Since any disturbance can be represented as a superposition of sinusoidal disturbances by the Fourier theorem, that means that all disturbances of sufficiently small amplitude are stable.

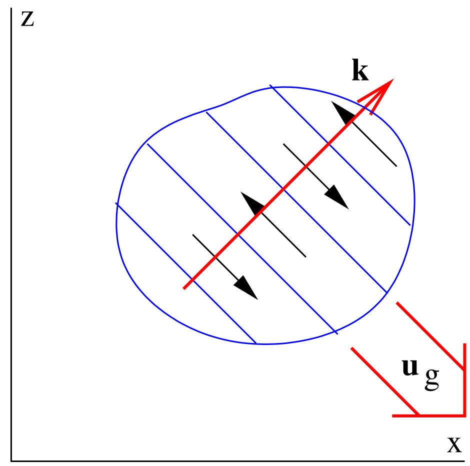

Figure 2.1: Gravity wave packet in its plane of oscillation, showing the wave vector $\mathbf{k}$, the group velocity $\mathbf{u}_g$, and the fluid motions.

The mass continuity equation leads us to the condition

$$\mathbf{k} \cdot \mathbf{v} = 0, \tag{2.19}$$

where $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector. Thus, the motion of fluid parcels is normal to the wave vector, which means that these waves are transverse. We can easily show that the group velocity $\mathbf{u}_g$ of the wave takes the form

$$u_{gx} = \frac{\partial \omega}{\partial k_x} = \frac{k_z^2 N}{(k_x^2 + k_z^2)^{3/2}}, \qquad u_{gz} = \frac{\partial \omega}{\partial k_z} = -\frac{k_x k_z N}{(k_x^2 + k_z^2)^{3/2}}, \tag{2.20}$$

from which we infer that the group velocity is also normal to the wave vector:

$$\mathbf{u}_g \cdot \mathbf{k} = 0. \tag{2.21}$$

Thus, a wave packet has the appearance in the $x - z$ plane as illustrated in figure 2.1. These waves are called *gravity waves*, and they are ubiquitous in the atmosphere.

From equation (2.18), we note that gravity waves have a maximum frequency equal to N when $k_z^2 \ll k_x^2$, i. e., when the wave fronts are vertical. The mechanism behind gravity waves becomes clear in this case. If the ambient buoyancy increases with height, then a lifted parcel finds itself in an environment with less buoyancy than its surroundings. Bouyancy forces thus

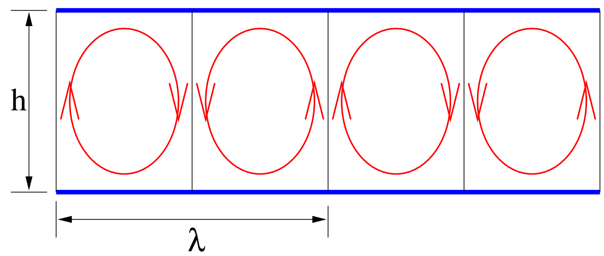

Figure 2.2: Flow pattern for convection, with $k_x = 2\pi/\lambda$ and $k_z = \pi/h$.

act to restore the parcel to its original level. The parcel overshoots, putting it into a region of positive buoyancy. The result is an oscillatory motion in which the parcel bobs up and down. Slantwise oscillations can also occur, but the frequency of such oscillations is reduced from N, as equation (2.18) shows. A typical oscillation period for vertical motions in the atmosphere is 10 min.

When $\gamma < 0$, we set $\gamma = -\sigma^2$ and write the dispersion relation

$$\omega = \pm \frac{ik_x\sigma}{(k_x^2 + k_z^2)^{1/2}}. \tag{2.22}$$

In this case, small perturbations to the horizontally homogeneous equilibrium state are unstable, since $\exp[i(k_x x + k_z z - \omega t)]$ grows with time if ω is positive imaginary. The resulting instability is called *convective instability*, and the fully developed flow is called *convection*. An alternative solution form is more physically appealing and gives rise to the same dispersion relation: $v_x \propto \cos(k_x x)\cos(k_z z)\exp(-i\omega t)$; $b', v_z \propto \sin(k_x x)\sin(k_x z)\exp(-i\omega t)$; $\pi' \propto \sin(k_x x)\cos(k_z z)\exp(-i\omega t)$. This flow pattern satisfies $v_z = 0$ at $z = 0, h$, where $h = \pi/k_z$. Thus, it can be thought of as the flow which occurs between two rigid surfaces at these levels. The resulting flow is shown in figure 2.2.

The largest growth rates in equation (2.22) occur when $k_z^2 \ll k_x^2$, i. e., when the convective cells are tall and skinny.

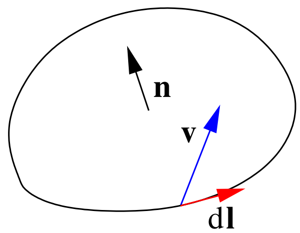

Figure 2.3: Illustration of a circulation loop with fluid velocity $\mathbf{v}$, unit normal $\mathbf{n}$ to the area bounded by the loop, and fluid flow velocity $\mathbf{v}$.

2.3 Kelvin circulation theorem

2.3.1 Derivation

The Kelvin circulation theorem expresses a result central to geophysical fluid dynamics. The *circulation* around a loop embedded in a fluid is defined

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l}. \tag{2.23}$$

The Kelvin theorem tells us how the circulation evolves with time under the condition that each element of the loop (illustrated in figure 2.3) moves with the fluid in which it is embedded. However, before stating and proving this theorem, we note that by Stokes' theorem, the circulation can also be written

$$\Gamma = \int \nabla \times \mathbf{v} \cdot \hat{\mathbf{n}} dA, \tag{2.24}$$

where the area integral is over any surface which is bounded by the circulation loop. The quantity $\zeta \equiv \nabla \times \mathbf{v}$ is called the *vorticity*.

In order to incorporate the fact that the circulation loop moves with the fluid, we approximate the integral in equation (2.23) by a finite sum:

$$\Gamma \approx \sum_i \mathbf{v}_i \cdot \Delta\mathbf{l}_i, \tag{2.25}$$

where $\Delta\mathbf{l}_i$ is the ith segment of the circulation loop and $\mathbf{v}_i$ is the fluid velocity in that segment. We assume that the segment moves with the fluid, so that $d\Delta\mathbf{l}_i/dt = \Delta\mathbf{v}_i$, i. e., the difference between the fluid velocities at each end of the loop segment. Thus,

$$\frac{d\Gamma}{dt} \approx \sum_i \left(\frac{d\mathbf{v}_i}{dt} \cdot \Delta\mathbf{l}_i + \mathbf{v}_i \cdot \Delta\mathbf{v}_i \right). \tag{2.26}$$

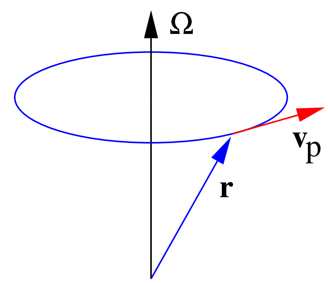

Figure 2.4: Planetary velocity $\mathbf{v}_p$ at position $\mathbf{r}$ for a planet with rotation rate and axis defined by the vector Ω.

Returning to the exact integral form and realizing that $\mathbf{v} \cdot d\mathbf{v} = d(v^2/2)$, this equation becomes

$$\frac{d\Gamma}{dt} = \oint \left(\frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} + d(v^2/2) \right). \tag{2.27}$$

The second term on the right side is a perfect differential which vanishes in a line integral around a closed loop.

The time derivative of the velocity is given by the momentum equation

$$\frac{d\mathbf{v}}{dt} = -\theta \nabla \Pi - \nabla \Phi, \tag{2.28}$$

where we have used the ideal gas form, neglecting for now both viscosity and rotation. The quantity $\Phi = gz$ is called the *geopotential*, and is the potential energy per unit mass associated with the gravitational and centrifugal forces. Except near boundaries or at very small scales, the molecular viscosity is unimportant in the atmosphere. We will reintroduce rotation shortly.

Introduction of equation (2.28) into (2.27) results in the geopotential term dropping out, since it also is in the form of a perfect differential: $\nabla\Phi \cdot d\mathbf{l} = d\Phi$. The net result is that

$$\frac{d\Gamma}{dt} = -\oint \theta d\Pi, \tag{2.29}$$

where we have used $\nabla\Pi \cdot d\mathbf{l} = d\Pi$.

Equation (2.29) is an expression of the Kelvin circulation theorem in an inertial reference frame in which viscous forces are negligible. To generalize this to the reference frame of the rotating earth, we note that the fluid velocity in the inertial frame is $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_r$, where $\mathbf{v}_p = \Omega \times \mathbf{r}$ is the planetary rotation velocity, as illustrated in figure 2.4, and $\mathbf{v}_r$ is the earth-relative velocity. The corresponding components of the circulation are $\Gamma = \Gamma_p + \Gamma_r$, where the Kelvin theorem as defined by equation (2.29) applies to Γ. To extend this to Γ_r, the circulation defined in the rotating reference frame of the earth, we note that

$$\Gamma_p = \oint \Omega \times \mathbf{r} \cdot d\mathbf{l} = \int [\nabla \times (\Omega \times \mathbf{r})] \cdot \hat{\mathbf{n}} dA = 2\Omega \cdot \int \hat{\mathbf{n}} dA, \tag{2.30}$$

from which we conclude that

$$\frac{d\Gamma_r}{dt} = \frac{d\Gamma}{dt} - \frac{d\Gamma_p}{dt} = -\oint \theta d\Pi - 2\Omega \cdot \frac{d}{dt} \int \hat{\mathbf{n}} dA. \tag{2.31}$$

An alternative form of this equation comes from eliminating Γ using equation (2.24) and bringing the second term on the right to the left side:

$$\frac{d}{dt} \int [\nabla \times \mathbf{v} + 2\Omega] \cdot \hat{\mathbf{n}} dA = -\oint \theta d\Pi. \tag{2.32}$$

The quantity $\zeta_a = \nabla \times \mathbf{v} + 2\Omega$ is called the *absolute vorticity*, and is in fact the vorticity as measured in a non-rotating reference frame. We refer to 2Ω as the *planetary vorticity*, as it is the part of the absolute vorticity which is caused by the earth's rotation.

2.3.2 Potential vorticity

Imagine a circulation loop embedded in a surface of constant potential temperature θ in an atmosphere in which $d\theta/dt = 0$. In this case the loop will always remain embedded in this surface, which by virtue of the conserved potential temperature is a material surface. In this case the right side of equation (2.32) is the integral over a closed loop of a perfect differential, and is therefore zero. If the constant potential temperature surface is flat (an assumption which becomes better as the circulation loop is made smaller), then equation (2.32) tells us that

$$\frac{d\zeta_{a\theta} A}{dt} = 0, \tag{2.33}$$

where $\zeta_{a\theta}$ is the component of absolute vorticity normal to the constant potential temperature surface and A is the area of the circulation loop. Except in extreme circumstances, isentropic surfaces are nearly horizontal, so $\zeta_{a\theta}$ is nearly the vertical component of the absolute vorticity. However, since

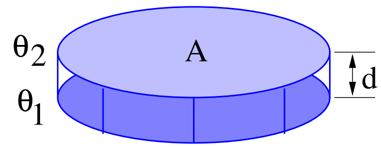

Figure 2.5: A control volume for the definition of potential vorticity. The upper and lower faces are surfaces of constant potential temperature.

the horizontal components of vorticity are generally much stronger than the vertical component, even a slight deviation from vertical of the normal to the isentropic surface can result in a significant contribution to $\zeta_{a\theta}$ from the horizontal components.

As A increases or decreases in response to the evolution of the flow, $\zeta_{a\theta}$ decreases or increases correspondingly. Furthermore, since

$$\zeta_{a\theta} = \zeta_\theta + \Omega_\theta \tag{2.34}$$

where ζ_θ and Ω_θ are the components of the relative vorticity and the earth's rotation vector normal to the isentropic surface, changes in latitude of a parcel result in a tradeoff between ζ_θ and Ω_θ.

Imagine now a control volume as shown in figure 2.5 with upper and lower faces being surfaces of constant potential temperature with area A and separation d. If the box moves and deforms with the flow, we can apply equation (2.33) to a path around the periphery of the volume. Since the surfaces bounding the volume advect with the fluid, the mass of fluid inside the volume M does not change with time. Furthermore, we have $M = \rho A d$, where ρ is the density of the fluid. Finally, we can relate d and $\theta_2 - \theta_1$ to the gradient of θ: $|\nabla\theta| = (\theta_2 - \theta_1)/d$. Putting these results together with equation (2.33), we conclude that

$$\frac{d}{dt}\left(\frac{M|\nabla\theta|\zeta_{a\theta}}{\rho(\theta_2 - \theta_1)}\right) = \frac{M}{(\theta_2 - \theta_1)}\frac{d}{dt}\left(\frac{\nabla\theta \cdot \zeta_a}{\rho}\right) = 0. \tag{2.35}$$

The quantity $q = \nabla\theta \cdot \zeta_a/\rho$ is called the *potential vorticity*, and we have shown that it is conserved in parcels,

$$\frac{dq}{dt} = 0, \tag{2.36}$$

when potential temperature is conserved and when no frictional forces operate. Later we will relax these restrictions. Potential vorticity is a powerful

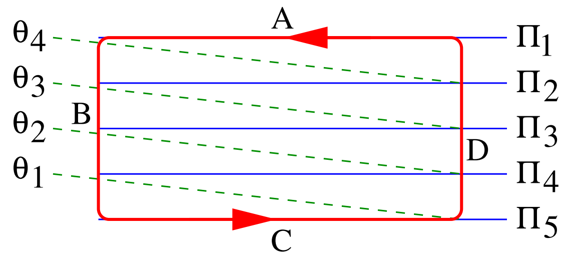

Figure 2.6: Circulation path illustrating baroclinic vorticity generation.

tool for understanding atmospheric dynamics in general, and the long-time-scale effect of convection on the environment in particular.

2.3.3 Baroclinic vorticity generation

Figure 2.6 indicates a vertically oriented circulation loop in a fluid with horizontal surfaces of constant Exner function, but tilted surfaces of constant potential temperature. Ignoring the horizontal component of planetary vorticity (which is generally small compared to the horizontal component of relative vorticity), we apply equation (2.29) to this situation. In the computation of the line integral around the illustrated path, the contribution from segments A and C is zero, since $d\Pi = 0$ there. Thus, the integral reduces to

$$\frac{d\Gamma}{dt} = -[\theta_B(\Pi_5 - \Pi_1) + \theta_D(\Pi_1 - \Pi_5)] = (\theta_D - \theta_B)(\Pi_5 - \Pi_1), \qquad (2.37)$$

where θ_B and θ_D are the average values of θ over the B and D segments of the line integral. Given the slope of the constant potential temperature surfaces (represented by the dashed lines) and the fact that θ increases upward in a stable atmosphere, we have $\theta_D > \theta_B$. Furthermore, $\Pi_5 > \Pi_1$, since Exner function decreases upward, so the circulation around the loop as defined in figure 2.6 increases with time. This suggests downward motion over segment B of the loop and upward motion over segment D, which produces increasing vorticity pointing out of the page according to Stokes' theorem. The resulting motion is in the sense needed to make the constant entropy surfaces horizontal.

2.4 Turbulence

Big whorls have little whorls
That feed on their velocity
And little whorls have lesser whorls
And so on to viscosity

L. F. Richardson (1922)

This ditty expresses an essential truth about turbulence, which is a three-dimensional chaotic flow in which many scales of motion are active, and in which a cascade of kinetic energy proceeds from larger to smaller scales, with viscosity converting the kinetic energy to heat at the smallest scales.

2.4.1 Reynolds decomposition

We first present a formalism which by itself contains no physics, but which represents a useful framework for discussing turbulence. Suppose we wish to compute numerically the evolution of a flow. This is normally done by approximating the continuous fluid equations by finite analogs in which dependent variables such as velocity and buoyancy are defined only on a regular grid. If the size of a grid cell is L, then structure and motion on scales less than L will not be represented explicitly in the calculation and must be represented parametrically. The division between the explicitly represented flow and the leftovers is called the *Reynolds decomposition*.

Let us define a low pass filter or smoothing operator by an overbar. This operator removes variance in velocity, buoyancy, and pressure on scales less then L while leaving scales greater than L untouched. A smoothed variable is fully representable by its values on the grid, whereas the part removed by the smoothing operator is not. In the context of the Boussinesq equations, we divide the velocity, buoyancy, and kinematic pressure into explicitly represented and implicit parts, the latter indicated by superscripted primes:

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}' \tag{2.38}$$

$$b = \overline{b} + b' \tag{2.39}$$

$$\pi = \overline{\pi} + \pi'. \tag{2.40}$$

We assume that $\overline{\mathbf{v}'} = 0$ and $\overline{\overline{\mathbf{v}}} = \overline{\mathbf{v}}$, that terms like $\overline{\overline{\mathbf{v}}\mathbf{v}'} \approx 0$, and that $\overline{\overline{\mathbf{v}}\,\overline{\mathbf{v}}} \approx \overline{\mathbf{v}}\,\overline{\mathbf{v}}$, etc.

Since $\nabla \cdot \mathbf{v} = 0$ in the Boussinesq case, we can write the Boussinesq equations for momentum and buoyancy in flux form as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{v}) + \nabla \pi - b\hat{\mathbf{z}} = 2\nu \nabla \cdot \mathbf{D} \tag{2.41}$$

$$\frac{\partial b}{\partial t} + \nabla \cdot (b\mathbf{v}) = \kappa \nabla^2 b \tag{2.42}$$

where we have ignored rotation. Substituting equations (2.38)-(2.40) into equations (2.41) and (2.42) and applying the smoothing operator results in

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + \nabla \cdot (\overline{\mathbf{v}}\,\overline{\mathbf{v}} + \overline{\mathbf{v}'\mathbf{v}'} - 2\nu\overline{\mathbf{D}}) + \nabla\overline{\pi} - \overline{b}\hat{\mathbf{z}} = 0 \tag{2.43}$$

$$\frac{\partial \overline{b}}{\partial t} + \nabla \cdot (\overline{b}\overline{\mathbf{v}} + \overline{b'\mathbf{v}'} - \kappa\nabla\overline{b}) = 0. \tag{2.44}$$

In addition we have for mass continuity

$$\nabla \cdot \overline{\mathbf{v}} = 0. \tag{2.45}$$

If we can now come up with models for the eddy correlation terms on the right sides of equations (2.43) and (2.44), then we have a closed set of approximate governing equations defined on a finite grid. The rest of this chapter is devoted to this quest.

The quantity $\overline{\mathbf{v}'\mathbf{v}'}$ is the subgrid scale flux of momentum, while $\overline{b'\mathbf{v}'}$ is the subgrid scale flux of buoyancy. In other words, these terms represent the transport of momentum and buoyancy by the small-scale motions which are not represented in the explicit flow calculation. A slightly different interpretation of the first term is as a type of stress called the *Reynolds stress*: $\mathbf{T}_R \equiv -\overline{\mathbf{v}'\mathbf{v}'}$. The divergence of the Reynolds stress acts like a force on the flow, but it is really due to the transport of momentum by subgrid scale eddies, just as the ordinary stress tensor is due to the transport of momentum by deviations in the motion of molecules from the bulk flow.

One immediate difficulty presents itself with the above formalism. The primed part of the flow is sometimes viewed as the turbulent part, whereas the smoothed part is taken to be some average or mean flow on which the turbulence is superimposed. In this case there is no guarantee that the grid size is the same as the characteristic size of the turbulent eddies. An alternate interpretation of the Reynolds decomposition is that it is not a decomposition based on spatial scale, but on the mean and variations within an ensemble of possible flows which are envisioned to be possible subject to a nearly fixed set of initial and boundary conditions. For instance, a thunderstorm which develops as a result of some specified set of meteorological forcings may have turbulent eddies which are changed significantly if the initial forcings are changed by even a tiny amount. However, the overall structure of the thunderstorm may be the same. If we imagine an ensemble of thunderstorms (think parallel universes) in which the forcings differ only by small amounts, then the variables with an overbar are ensemble averages, whereas the overbars of products of primed variables are averages over deviations from the ensemble mean.

The ensemble average form of the Reynolds decomposition has the advantage that the primed part of the flow can be related to real physical processes, whereas for the scale separation form this cannot be so easily done. However, real world measurements are harder to interpret in the context of the ensemble average form, as we (currently) cannot make measurements in parallel universes! The best we can do is try to find similar instances of a phenomenon in the one universe to which we have access and approximate the ensemble average as an average over these instances.

2.4.2 Inertial subrange

Turbulence generally forms as the result of the breakdown of a smooth flow due to an instability. The spatial scale of the instability L is called the *outer scale* of the turbulence. Energy is supplied to the turbulence at this scale. In turbulence, we envision eddies successively breaking down into smaller eddies, which in turn decay into even smaller eddies, as expressed above by Richardson. A lower limit exists for this breakdown when viscous or other diffusive processes take over from bulk motion in mixing fluid properties. The eddy size λ at which this occurs is called the *inner scale*. Between these scales, no kinetic energy is lost or gained; it simply cascades to successively smaller scales from the outer scale to the inner scale. This range of scales is called the *inertial subrange*, since only inertial forces are important in this range.

Eddies are thought to decay to smaller eddies after turning over once. If an eddy of diameter l has a typical velocity v_l, then the time for turnover is approximately $t_l = l/v_l$. The kinetic energy per unit volume at scale l (ignoring the factor of two) is v_l^2, so the rate at which kinetic energy cascades from one scale to the next is

$$\epsilon = v_l^2/t_l = v_l^3/l. \tag{2.46}$$

Since we assume that kinetic energy is not lost as it cascades to smaller scales, the *dissipation rate* ϵ must be independent of scale within the inertial subrange. This leads to an estimate for eddy velocity as a function of scale:

$$v_l = (l\epsilon)^{1/3}. \tag{2.47}$$

The breakup of eddies into smaller eddies actually increases the vorticity as one moves to smaller scales. This may be seen by noting that the vorticity scales as

$$\zeta_l = v_l/l = \frac{\epsilon^{1/3}}{l^{2/3}}. \tag{2.48}$$

So even though the characteristic eddy velocity decreases as one goes to smaller scales, the characteristic vorticity increases. This can only happen (in the absence of other forces) via vortex stretching, which implies that true turbulence is always three-dimensional. Purely two-dimensional flows cannot change the vorticity of parcels in the absence of external forces, which means that there is no such thing as two-dimensional turbulence!

The inner scale is reached when viscous stresses, which scale as $\nu v_l/l$, become comparable to Reynolds stresses, which scale as v_l^2. The kinematic viscosity is denoted by ν. Equating these two, setting $l = \lambda$, and solving for λ gives us an estimate of the inner scale:

$$\lambda = \frac{\nu^{3/4}}{\epsilon^{1/4}}. \tag{2.49}$$

At the outer scale, the order of magnitude of the nonlinear advection term in the momentum equation $\mathbf{v} \cdot \nabla \mathbf{v}$ divided by the order of magnitude of the viscosity term $\nu \nabla^2 \mathbf{v}$ is called the Reynolds number:

$$\mathrm{Re} = \frac{v_L L}{\nu} = \frac{L^{4/3}\epsilon^{1/3}}{\nu}. \tag{2.50}$$

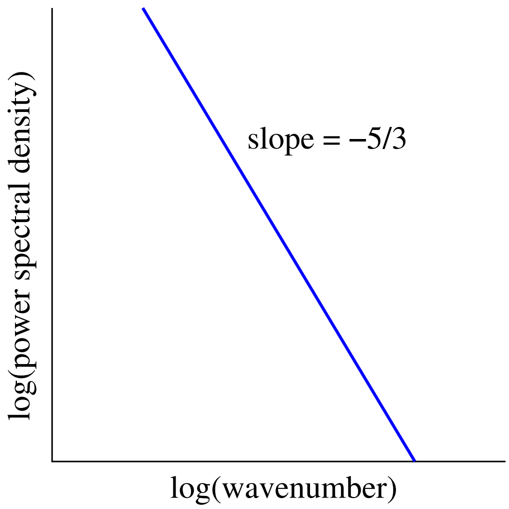

Figure 2.7: Power spectral density for inertial subrange.

2.4.3 Power spectrum

A Fourier analysis of the kinetic energy per unit mass in the inertial subrange of turbulence yields a *power spectrum* of the form seen in figure 2.7, i. e., a straight line in log-log space with a slope of $-5/3$. This represents the famous *minus five-thirds law*, and we now endeavor to explain it in terms of what we have learned.

We approximate wavenumber by the inverse of the length scale, $k = 1/l$. In our picture of eddies giving rise to smaller eddies, we imagine typical eddies to be half the size of their predecessor eddies. The succession of decreasing eddy sizes is thus a geometric rather than an arithmetic series in increasing wavenumber. From equation (2.47), the typical velocity for eddies of wavenumber k is $v_k = (\epsilon/k)^{1/3}$, which means that the kinetic energy per unit mass scales as $(\epsilon/k)^{2/3}$. Since a geometric series in wavenumber, i. e., $k = 1/L,\ 2/L,\ 4/L,\ 8/L,\ ...$) where L is the outer scale, implies a constant interval in $\log(k)$ rather than in k, each eddy generation contributes $(\epsilon/k)^{2/3}d\log(k) = (\epsilon/k)^{2/3}dk/k = \epsilon^{2/3}k^{-5/3}dk$ to the power spectrum, whence the $-5/3$ slope in the log-log plot of power spectral density.

2.4.4 Buoyant generation of turbulence

Convective instability can serve as an energy source for turbulence. The buoyancy perturbation b' of a parcel (actual minus environmental value) gives us the buoyant force per unit mass acting on the parcel. The work done on the parcel under a vertical displacement δz is therefore $b'\delta z$. If the z derivative of the environmental buoyancy profile is γ, then the buoyancy perturbation as a function of displacement δz is $-\gamma\delta z$, so the work done in a vertical displacement L on a parcel starting from buoyancy equilibrium is $-\gamma L^2/2$. Equating this to the kinetic energy and solving for the velocity, we get an estimate of the eddy velocity at the outer scale L where we ignore numerical factors of order unity:

$$v_L = (-\gamma)^{1/2} L. \tag{2.51}$$

(Recall that convective instability occurs when $\gamma < 0$.) This sets the dissipation rate in the turbulent cascade produced by the convective instability:

$$\epsilon = v_L^3/L = (-\gamma)^{3/2} L^2. \tag{2.52}$$

We now demonstrate that the kinetic energy produced by buoyancy at scales much smaller than the outer scale is negligible in comparison to the kinetic energy cascaded from larger scales. By simply repeating the above arguments, we see that this specific kinetic energy production rate at scale l is $(-\gamma)^{3/2} l^2$, which is less than ϵ by the factor $(l/L)^2$. Thus, to a good approximation the energy dissipation rate is independent of scale and the cascade obeys the laws of the inertial subrange.

2.4.5 Shear instability

Just because the atmosphere is stably stratified does not mean that instability cannot occur. If the atmosphere is *sheared*, i. e., if the horizontal wind varies with height, a parcel displaced vertically tends to retain the horizontal velocity of its original level, and thus acquire a velocity different from its new surroundings. Reference to figure 2.8 shows that the difference between the parcel velocity and the velocity of its new surroundings is $v' = -S\delta z$, where the shear is $S = \partial v/\partial z$ and where δz is the vertical displacement of the parcel. The parcel thus has a specific kinetic energy relative to its new surroundings of $S^2\delta z^2/2$. This compares to the energy required to lift it

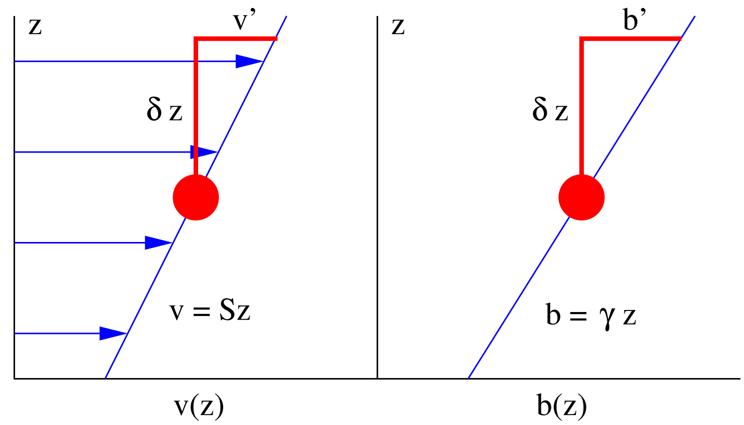

Figure 2.8: Results of parcel displacement in a sheared, stably stratified environment.

this distance against the buoyancy force $\gamma\delta z^2/2$. (We have retained the numerical factors in this calculation.) One might reasonably expect instability to be possible if the available kinetic energy exceeds the energy required to overcome buoyancy: $S^2\delta z^2/2 > \gamma\delta z^2/2$, or

$$\mathrm{Ri} = \frac{\gamma}{S^2} < 1. \tag{2.53}$$

The dimensionless quantity Ri is called the *Richardson number*. (Recall that Richardson is the author of the ditty about turbulence quoted at the beginning of this section!) Though this argument is rather crude, the above condition serves as a reasonable approximate guide to the instability of sheared flows.

Though buoyancy serves to extract kinetic energy from the flow in this case rather than add to it, turbulence generated by this *shear instability* exhibits an inertial subrange for the same reasons that convective instability does. The outer scale of the turbulence is generally determined by the vertical thickness of the unstable shear layer.

2.4.6 Mixing length theory

The German physicist Ludwig Prandtl, the father of aerodynamics, came up with a theory of turbulence which stands to this day. Suppose we wish to estimate the turbulent buoyancy flux $\overline{\mathbf{v}'b'}$. Prandtl visualized turbulent flows as consisting of parcels which move from their original location a distance l at speed v before mixing with their new surroundings. The quantity v is an estimate of the magnitude of the turbulent velocity $\mathbf{v}'$ while l is called the *Prandtl mixing length*. If a gradient $\nabla\bar{b}$ in the mean buoyancy exists, then the buoyancy perturbation b' in the turbulent parcel will be of order $l|\nabla\bar{b}|$. Assuming that buoyancy flows down the buoyancy gradient, then we postulate on the basis of the above arguments that

$$\overline{\mathbf{v}'b'} = -Clv\nabla\bar{b} \equiv -K\nabla\bar{b} \tag{2.54}$$

where C is a dimensionless constant and $K = Clv$ is the *eddy mixing coefficient*.

Similar arguments applied to the Reynolds stress would seem to suggest that $\overline{\mathbf{v}'\mathbf{v}'} = -K\nabla\bar{\mathbf{v}}$, but this expression cannot be correct because the left side is a symmetric tensor while the right side is the deformation rate tensor which has no defined symmetry. However, if we go by analogy with the molecular stress tensor in the case of an incompressible fluid, then we can write

$$\overline{\mathbf{v}'\mathbf{v}'} = -2K\mathbf{D} \tag{2.55}$$

where $\mathbf{D}$ is the strain rate tensor. This suggests that the rotation rate tensor $\mathbf{R}$ plays no role in the generation of turbulence.

The determination of the mixing length l and the characteristic velocity v of turbulent eddies is a matter for theories of turbulent flow in each situation to decide. Since turbulence typically has eddies with all sizes between the inner and outer scales, all scales should contribute to the determination of the eddy mixing coefficient. However, equation (2.47) shows that the eddy turnover velocity in the inertial subrange is roughly $(l\epsilon)^{1/3}$, so the contribution to the eddy mixing coefficient for length scale l is of order $l^{4/3}\epsilon^{1/3}$. This increases with eddy size, which means that since ϵ is constant in the inertial subrange, the biggest contribution to K comes from eddies on the largest scale, i. e., at the outer scale for the turbulence. Thus identifying l in the definition of eddy mixing coefficient with the outer scale is justified.

In three-dimensional numerical calculations on a finite grid, these ideas can best be applied if the grid size is small enough to put subgrid scale turbulent eddies in the inertial subrange. In this case the largest turbulent eddies are assumed to be computed explicitly by the model, and the parameterization of subgrid scale flow need not take into account the mechanisms generating the turbulence – all inertial subrange turbulence is assumed to have the same characteristics. The mixing length l for the turbulence is then assumed to be the grid size L and the characteristic turbulent velocity is taken to be proportional to the absolute value of the strain rate times L:

$$K = CL^2|\mathbf{D}|. \tag{2.56}$$

The absolute value of the strain rate is the square root of the sum of the squares of the terms and $C \approx 0.2$ in many applications.

In stably stratified shear flow for which the Richardson number $\mathrm{Ri} > 1$, generally no turbulence is found, as indicated above. This is a common situation in the atmosphere, and in this case equation (2.56) is often modified in numerical models by adding a Richardson number dependence which zeros K when $\mathrm{Ri} > 1$.

2.5 References

Landahl, M. T., and E. Mollo-Christensen, 1986: *Turbulence and Random Processes in Fluid Mechanics.* Cambridge University Press, Cambridge. A short and not too-exotic treatment of turbulence.

Pedlosky, J., 2003: *Waves in the Ocean and Atmosphere.* Springer-Verlag, Berlin. A relatively recent treatment of waves in geophysical fluids.

2.6 Problems

1. Using the methods of section 2.1 derive an equation for $b^2/2$ which takes the variable/flux/source form of equation (2.12). Hint: Multiply equation (2.4) by b.

2. Show that the integral of equation (2.12) over some volume tells us that the time rate of change of energy in the volume equals the volume integral of the source term minus the area integral of the flux of energy out of the volume.

3. Do a stability analysis for a stratified, inviscid, non-diffusive Boussinesq fluid at rest in a rotating frame with the rotation vector Ω pointing vertically upward. Examine what happens for both positive and negative (but constant) $\gamma = db_0/dz$.

4. Show that the mean value of the vertical component of the absolute vorticity averaged globally over a constant height surface in the earth's atmosphere is zero. (Hint: Think of Gauss' law for magnetism; what property does vorticity share with the magnetic field?)

5. Compute the time scale for eddy overturning as a function of eddy size l in the inertial subrange. From this, comment on the rapidity with which smaller scales adjust to changes in conditions at the outer scale.

6. Law of the wall. Consider turbulent, non-buoyant ($b = 0$) fluid with a time-independent mean flow in the x direction $v_x(z)$ parallel to a rigid wall at $z = 0$. Ignore viscosity initially and assume that the mean kinetic pressure $\pi = 0$.

   (a) The only length available to determine the size of the dominant (i. e., outer scale) turbulent eddies a distance z from the wall is z itself. Write an expression for the eddy mixing coefficient in terms of z and $v_x(z)$ using Prandtl mixing length theory.

   (b) Determine $v_x(z)$ assuming that the turbulent eddy flux of x momentum in the z direction is (a uniform) $-T_{Rxz}$ and the viscous flux is negligible. (Recall that $\mathbf{T}_R$ is the Reynolds stress. Why must this be independent of z?)

   (c) Is this solution realistic as $z \to 0$? Explain what happens physically there, and estimate the distance from the wall z_c that the solution you have obtained breaks down. Hint: At what distance from the wall does the Reynolds stress T_{Rxz} equal the viscous stress T_{xz}?

   (d) Find $v_x(z)$ close to the wall where the viscous stress dominates.