Chapter 1

Principle of virtual work

1.1 Constraints and degrees of freedom

The number of degrees of freedom of a system is equal to the number of variables required to describe the state of the system. For instance:

- A particle constrained to move along the x axis has one degree of freedom, the position x on this axis.
- A particle constrained to the surface of the earth has two degrees of freedom, longitude and latitude.
- A wheel rotating on a fixed axle has one degree of freedom, the angle of rotation.
- A solid body in free space has six degrees of freedom: a particular atom in the body can move in three dimensions, which accounts for three degrees of freedom; another atom can move on a sphere with the first particle at its center for two additional degrees of freedom; any other atom can move in a circle about the line passing through the first two atoms. No other independent motion of the body is possible.
- N atoms moving freely in three-dimensional space collectively have $3N$ degrees of freedom.

1.1.1 Holonomic constraints

Suppose a mass is constrained to move in a circle of radius R in the x-y plane. Without this constraint it could move freely over this plane. Such a constraint could be expressed by the equation for a circle, $x^2 + y^2 = R^2$. A better way to represent this constraint is

$$F(x, y) = x^2 + y^2 - R^2 = 0. \tag{1.1.1}$$

As we shall see, this constraint may be useful when expressed in differential form:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 2xdx + 2ydy = 0. \tag{1.1.2}$$

A constraint that can be represented by setting to zero a function of the variables representing the configuration of a system (e.g., the x and y locations of a mass moving in a plane) is called *holonomic*.

In a more complex system, there may be more than one constraint. For instance, if the mass in the above case is moving in x-y-z space, but in addition is constrained to remain on a horizontal surface at elevation a, the additional constraint

$$G(z) = z - a = 0 \tag{1.1.3}$$

would apply.

1.1.2 Non-holonomic constraints

Sometimes a constraint on the motion of an object cannot be represented in holonomic form. For instance, imagine a car moving on a horizontal plane. The car would normally have three degrees of freedom, two translational represented by the car's position (x, y) and a rotational degree of freedom about the vertical axis, represented by an angle ϕ counterclockwise from the x axis. (We neglect the possibility of the car overturning!) However, if the car is not skidding, it is constrained at a particular instant to move in the direction it is pointing, which can be represented by the differential relation

$$\sin\phi dx - \cos\phi dy = 0. \tag{1.1.4}$$

This constraint cannot be integrated to the form $F(x, y) = 0$, because ϕ can change as the car moves due to the driver turning the steering wheel. It thus depends on more than x and y. Thus, for infinitesimal motions, the car can only move along a particular line in the x-y plane as represented by equation (1.1.4), whereas with driver input, the car can reach any point in this plane with any rotational orientation, but only through finite motions. (This is what makes parallel parking so complicated!) Thus a car exhibits only one degree of freedom in infinitesimal motion, but three degrees of freedom in finite motion.

A constraint of this type is called *non-holonomic*. In general, non-holonomic constraints are more difficult to deal with than holonomic constraints.

1.2 Internal and external forces on a system

In mechanics, the definition of what constitutes a *system* is arbitrary; the choice is completely up to us, and is based on what we are trying to accomplish. A system consisting of many atoms is in principle very complicated, because one must consider not only external forces

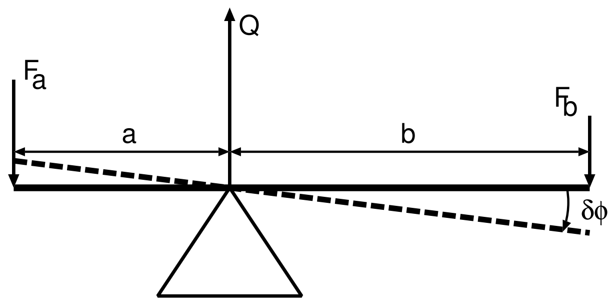

Figure 1.3.1: The balance beam.

acting on the system, but internal forces acting between each pair of atoms within the system. However, Newton's third law says that the force of atom A on atom B is equal and opposite to the force of atom B on atom A. Thus, the net force on the system due to atoms within the system acting on each other is zero. This result is related to the conservation of linear momentum in isolated systems. Thus, in considering the overall motion of a system, only external forces need be considered.

1.3 Principle of virtual work

The modern approach to a statics problem is to apply the two conditions that the total force and the total torque acting the system of interest each sum to zero. Sommerfeld invokes an older method of handling such problems called the *principle of virtual work*. This method has the advantage that forces of constraint, i.e., forces that keep the system from moving, may be neglected, thus potentially simplifying the analysis. Only forces, not torques, need to be considered, as the locations at which the forces are applied are used in the analysis. Since the action of a torque is really the action of a force applied at a particular location, the consideration of torques becomes less important.

1.3.1 Uneven balance beam

A simple example is the uneven balance beam, illustrated in figure 1.3.1. The modern approach sets the total force and torque on the beam to zero:

$$Q - F_a - F_b = 0 \tag{1.3.1}$$

$$aF_a - bF_b = 0. \tag{1.3.2}$$

Solving the first equation tells us that the upward force of the pivot on the beam just balances the two downward forces at the opposite ends of the beam:

$$Q = F_a + F_b. \tag{1.3.3}$$

The second equation gives us the ratio of the two end forces:

$$\frac{F_a}{F_b} = \frac{b}{a}. \tag{1.3.4}$$

These forces can also be obtained in terms of the beam dimensions and Q:

$$F_a = \frac{bQ}{a+b} \qquad F_b = \frac{aQ}{a+b} \tag{1.3.5}$$

The principle of virtual work can be used to obtain the same results. The idea is that if the beam tilts by a small angle $\delta\phi$ in the clockwise direction, then the forces F_a and F_b respectively do work $-F_a a\delta\phi$ and $F_b b\delta\phi$. This is because the left end of the beam moves a distance $a\delta\phi$ in a direction opposite that of the force whereas the right end of the beam moves a distance $b\delta\phi$ in the same direction as the force. The force of the pivot on the beam does no work, as the pivot is assumed not to move, so the total work done is

$$\delta W = (-F_a a + F_b b)\delta\phi. \tag{1.3.6}$$

This work increment is zero as a result of the torque balance expressed by equation (1.3.2). Thus, assuming that the work increment is zero in a small displacement of the system is equivalent to the condition of zero net torque.

If instead of tilting the beam, the pivot and beam are lifted vertically by a small distance δz, the work done by the three forces in this case would be

$$\delta W = (Q - F_a - F_b)\delta z, \tag{1.3.7}$$

which according to equation (1.3.1) is also zero. Setting $\delta W = 0$ in this equation allows us to obtain the pivot support force Q. Thus, setting the work done in small displacements of the system to zero allows us to determine all of the relevant forces via computing the work done by these forces. The work is called *virtual*, because no real motion of the system is envisioned, with the corresponding complications resulting from generation of kinetic energy.

In this example, there is little computational advantage in using the principle of virtual work over the method of zero forces and torques. However, its virtues become more evident as the problem becomes more complex.

1.3.2 Ladder

Let's now consider the ladder problem illustrated in figure 1.3.2. A person of mass M is standing part way up a ladder leaning against a frictionless wall. The ladder is kept in place by frictional force F acting on the bottom of the ladder as illustrated. This is a problem with one degree of freedom, reflected by potential changes in the angle ϕ of the ladder relative to the wall. Two forces do work when a small displacement is made in the tilt of the ladder, gravity on the person standing on the ladder and the frictional force holding the ladder in place. (The mass of the ladder itself is assumed to be negligible.)

The person is a height $z = b\cos\phi$ above the floor, so a change in ϕ results in a change in height $\delta z = -b\sin\phi\delta\phi$. The work done by gravity on the person-ladder system with such a change is therefore $\delta W_g = gMb\sin\phi\delta\phi$. The distance of the base of the ladder from the wall is

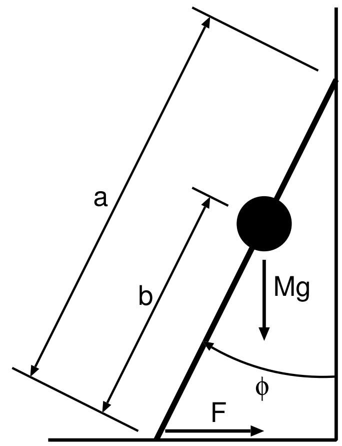

Figure 1.3.2: Ladder problem.

$x = a \sin\phi$, so the work done by the frictional force F is $\delta W_F = -F\delta(a\sin\phi) = -Fa\cos\phi\delta\phi$. Setting the total work to zero results in

$$(gMb\sin\phi - Fa\cos\phi)\delta\phi = 0. \tag{1.3.8}$$

Thus,

$$F = \frac{gMb\tan\phi}{a}. \tag{1.3.9}$$

Both of the above examples illustrate an important aspect of the method of virtual work. Forces of constraint acting on the system that do no work need not be considered. These include the support force of the pivot in the uneven balance beam and the normal forces of the floor and the wall on the ladder. However, these forces can be determined by making small virtual displacements of the system that cause the forces of constraint to do work, as illustrated in the determination of the pivot force in the balance beam problem.

1.3.3 Motion of piston and crankshaft

We now consider a problem with somewhat more complex geometry, namely the transmission of the force on a piston due to the combustion of a fuel-air mixture to the crankshaft of a reciprocating engine such as exists in most automobiles. We take this as a statics problem on the system that includes the piston and the connecting rod of length L. The hot gas exerts a total pressure force P on the face of the piston. The piston exerts a force on the connecting rod, but the force of the connecting rod on the piston is equal and opposite, so no net work is done by these forces. However, the crankshaft arm is outside the system, so the component of the force normal to the arm F that the arm exerts on the connecting rod does work on the system. However, the component of this force parallel to the arm does not.

The geometry of the triangle with the three small circles as vertices in figure 1.3.3 must be analyzed. By the law of sines,

$$\sin\phi = \frac{R}{L}\sin\theta, \tag{1.3.10}$$

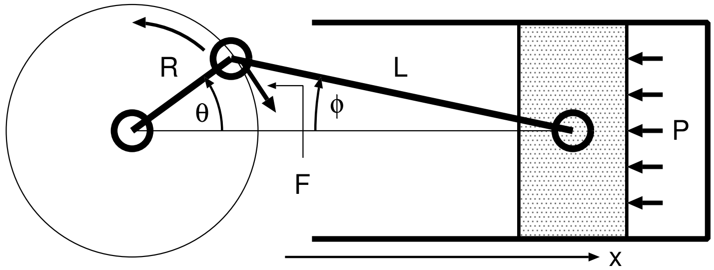

Figure 1.3.3: Motion of a piston and crankshaft in an automobile engine.

so

$$\delta\phi = \frac{R\cos\theta}{L\cos\phi}\delta\theta. \tag{1.3.11}$$

The x position of the face of the piston is given by

$$x = R\cos\theta + L\cos\phi + constant \tag{1.3.12}$$

where the value of the constant is immaterial, since we now take the differential of this equation:

$$\delta x = -R\sin\theta\delta\theta - L\sin\phi\delta\phi. \tag{1.3.13}$$

Using equations (1.3.10) and (1.3.11), this becomes

$$\delta x = -\left[\sin\theta + \frac{R\cos\theta\sin\theta}{\left(R^2 - L^2\sin^2\theta\right)^{1/2}}\right]R\delta\theta. \tag{1.3.14}$$

The work done on the system (i.e., the piston and connecting rod) by the pressure on the piston in a small displacement δx is $\delta W_P = -P\delta x$, whereas the work done by the crankshaft on the connecting rod with a small crankshaft rotation $\delta\theta$ is $\delta W_C = -FR\delta\theta$. Combining these and eliminating δx in favor of $\delta\theta$ results in total work on the system of

$$\begin{aligned}\delta W =& \delta W_C + \delta W_P = -FR\delta\theta - P\delta x \\ =& \left\{-F + \left[\sin\theta + \frac{R\cos\theta\sin\theta}{\left(R^2 - L^2\sin^2\theta\right)^{1/2}}\right]P\right\}R\delta\theta = 0,\end{aligned} \tag{1.3.15}$$

which is zero by the principle of virtual work. Thus, the force of the crankshaft arm on the connecting rod is

$$F = \left[\sin\theta + \frac{R\cos\theta\sin\theta}{\left(R^2 - L^2\sin^2\theta\right)^{1/2}}\right]P. \tag{1.3.16}$$

By Newton's third law, the force of the connecting rod on the crankshaft arm in the sense of positive $\delta\theta$ is also F, and the torque on the crankshaft is FR.

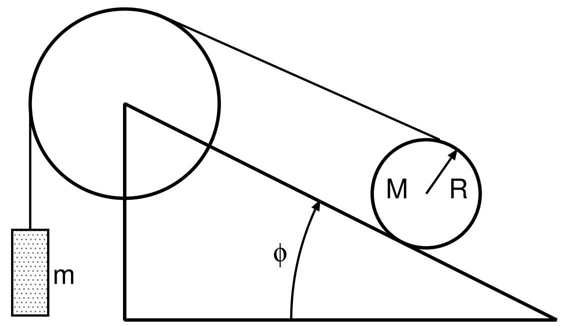

Figure 1.4.1: The pulley on the left is frictionless and the wheel on the right rolls up or down the ramp. The string wraps around the wheel on the ramp.

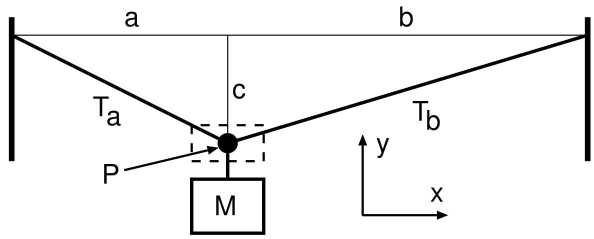

Figure 1.4.2: Mass hanging from clothes line.

1.4 Problems

1. Use the principle of virtual work to determine the ratio M/m that results in static equilibrium in figure 1.4.1.

2. Use the principle of virtual work to determine the tensions in the clothes line from which a mass M is hung, as illustrated in figure 1.4.2. Note that you will need to apply the principle twice, once for small virtual displacements of the point P horizontally and again for small vertical displacements. The tensions T_a and T_b are kept constant in these displacements.