Chapter 1

Solid body motion

In this chapter we extend what we have learned about the two-dimensional motion of solid bodies to three dimensions. We first show that any combination of translations and rotations of a solid can be represented as a displacement of some point in the solid plus a single rotation about some axis passing through this point. On this basis we calculate the angular momentum and the kinetic energy of rotation of a solid, introducing the moment of inertia tensor in the process. After this, a number of examples are considered.

1.1 Displacement and rotation of a solid

Suppose we pick three points in a solid body, O, A, and B. In the arbitrary movement of the body to a new location with a new orientation, the point O will undergo a displacement. Furthermore, the points A and B will be relocated relative to O. Let us call the locations relative to O of these relocated points A' and B', as illustrated in figure 1.1.

The fact that these points are all fixed in a solid body imposes certain constraints. In particular, the distances OA and OA' must be the same, which reduces the three degrees of

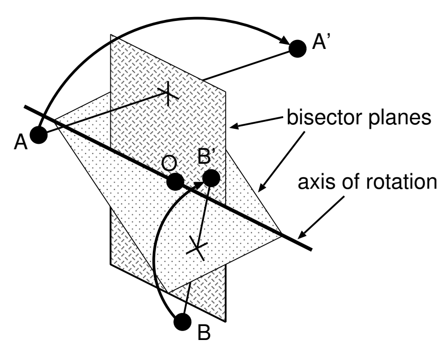

Figure 1.1: Finding the axis of rotation of a solid. Points A and B are shown relative to point O before the arbitrary displacement of the solid. Points A' and B' are their location relative to O after the displacement.

freedom of this point to two. Second, OB must equal OB' for the same reason. In addition, the distances AB and $A'B'$ must also be equal. These two conditions on point B reduce its degrees of freedom to one. Thus, the relocation of these three points accounts for three translational degrees of freedom in the displacement of point O, two rotational degrees of freedom in the displacement of point A relative to point O, and one rotational degree of freedom in the displacement of point B relative to points O and A. The total of six accounts for all of the degrees of freedom of a three-dimensional solid body, so by specifying these displacements, we have completely specified the translation and reorientation of the body.

To begin, we note that the locus of all points equidistant from A and A' is a plane normal to the line connecting A and A'. Furthermore, this plane must intersect the line at its midpoint. Let us call this the bisector plane for A and A'. A similar bisector plane may be defined for B and B'.

In solid body rotation each point in the body maintains a constant distance from the rotation axis. Thus, the rotation axis must lie in each of the bisector planes discussed above and is therefore defined by their intersection. Furthermore, the point O must also lie in this intersection.

Figure 1.1 illustrates the points considered as well as the bisector planes. The arcs connecting A, A' and B, B' are also shown, and illustrate the paths followed by these two points in the rotation. The rotation angles are the same for both of these points. Since all degrees of freedom have been accounted for, any other point in the solid must execute a similar arc as the solid body rotates to its new orientation. Therefore, we have shown that the arbitrary displacement of a solid body may be represented by the displacement of a point in the body followed by a rotation about this point, and we have shown how to construct the rotation axis.

We now consider infinitesimal rotations with rotation angle $d\phi$. For a mass element in the solid dm starting at location $\boldsymbol{r}$ relative to some point on the rotation axis, the change in position of the of the element is

$$d\boldsymbol{r} = \boldsymbol{n} \times \boldsymbol{r} d\phi \tag{1.1}$$

where $\boldsymbol{n}$ is a unit vector parallel to the axis of rotation. Defining dt as the time for this rotation to take place, the velocity of the mass element is

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{n} \times \boldsymbol{r}\frac{d\phi}{dt} = \boldsymbol{\Omega} \times \boldsymbol{r} \tag{1.2}$$

where $\boldsymbol{\Omega} = \boldsymbol{n}(d\phi/dt)$ is the rotation rate vector, the magnitude of which is the rotation rate and the direction specifies the orientation of the rotation axis. Recall further that the position of the rotation axis is important, as the tail of $\boldsymbol{r}$ lies on it. (Any point on the axis will do.)

1.2 Angular momentum and kinetic energy

Let us use the above result to calculate the angular momentum and kinetic energy of a solid body. Integrating over all mass elements in the body, the angular momentum is defined as

$$\boldsymbol{L} = \int \boldsymbol{r} \times \boldsymbol{v} dm = \int \boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) dm. \tag{1.3}$$

Using standard vector analysis, the integrand can be reduced to

$$\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega}(\boldsymbol{r} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\Omega} \cdot \boldsymbol{r}). \tag{1.4}$$

Evaluating term by term, the components of this double cross product can be represented in matrix form as

$$\begin{pmatrix} (y^2+z^2)\Omega_x - xy\Omega_y - xz\Omega_z \\ -xy\Omega_x + (x^2+z^2)\Omega_y - yz\Omega_z \\ -xz\Omega_x - yz\Omega_y + (x^2+y^2)\Omega_z \end{pmatrix} = \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}. \tag{1.5}$$

With this form, the rotation vector can be drawn out of the integral, and the angular momentum may be written

$$\boldsymbol{L} = \boldsymbol{I} \cdot \boldsymbol{\Omega} \tag{1.6}$$

where

$$\boldsymbol{I} = \int \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dm \tag{1.7}$$

is called the inertia tensor.

The kinetic energy of a solid in motion is

$$\begin{aligned} T =& \frac{1}{2} \int \boldsymbol{v} \cdot \boldsymbol{v} dm \\ =& \frac{1}{2} \int (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) dm \\ =& \frac{1}{2} \boldsymbol{\Omega} \cdot \int \boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) dm \\ =& \frac{1}{2} \boldsymbol{\Omega} \cdot \boldsymbol{L} \\ =& \frac{1}{2} \boldsymbol{\Omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\Omega} \end{aligned} \tag{1.8}$$

where line 3 derives from the vector identity $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. Thus the kinetic energy can also be expressed in terms of the rotation vector and inertia tensor.

The inertia tensor is symmetric and it therefore has real eigenvalues and orthogonal eigenvectors, the latter of which, when normalized, form the rows of the orthogonal matrix $\boldsymbol{U}$ that transforms vectors and tensors to the principal axis reference frame. If the eigenvalues of the inertia tensor are (I_1, I_2, I_3) and the components of the rotation vector and angular momentum in the principal axis frame are $(\Omega_1, \Omega_2, \Omega_3)$ and (L_1, L_2, L_3), then we have

$$L_1 = I_1\Omega_1 \qquad L_2 = I_2\Omega_2 \qquad L_3 = I_3\Omega_3 \tag{1.9}$$

and

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \tag{1.10}$$

in this frame.

1.3 So, what is a tensor???

You may be wondering what a tensor is at this point. We start by reminding ourselves that a vector is a quantity with direction and magnitude. For purposes of calculation, we express a vector in terms of its Cartesian components, say, $\boldsymbol{A} = (A_x, A_y, A_z)$. However, a set of three numbers is not itself a vector; we have to know in which coordinate system these components are expressed and we need to be able to get the components in a different coordinate system.

Let us consider the following matrix:

$$\boldsymbol{U} = \begin{pmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{pmatrix} \tag{1.11}$$

suppose that (U_{xx}, U_{xy}, U_{xz}) form the x unit vector in a new reference frame (call it the primed frame) expressed in terms of the components in the old reference frame (call it the unprimed frame). The second and third rows are the y and z unit vectors respectively. Thus, the components in the primed frame of the vector $\boldsymbol{A}$ are given by

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1.12}$$

The components of the $\boldsymbol{U}$ matrix take the form

$$U_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j \tag{1.13}$$

where $\boldsymbol{e}'_i$ is the ith unit vector in the primed frame and $\boldsymbol{e}_j$ is the jth unit vector in the unprimed frame. All this makes sense if you multiply out the matrix product in equation (1.12), getting three equations for the three components of $\boldsymbol{A}$ in the primed frame. Try it! It is also evident that the transpose of $\boldsymbol{U}$, denoted $\boldsymbol{U}^T$, transforms the primed form of $\boldsymbol{A}$ back to the unprimed form. Matrices of type $\boldsymbol{U}$ are called *orthogonal* matrices and we refer $\boldsymbol{U}$ to a *rotation matrix*. Orthogonal matrices also have the property that the transpose is also the inverse: $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$, so that $\boldsymbol{U} \cdot \boldsymbol{U}^T = \boldsymbol{U}^T \cdot \boldsymbol{U} = \boldsymbol{J}$, the identity matrix, i.e., the matrix with 1 on the diagonal and 0 off the diagonal. (We use $\boldsymbol{J}$ rather than $\boldsymbol{I}$ so as not to confuse the identity matrix with the inertia tensor.)

So, what about tensors? The physical definition of a tensor, analogous to the definition of a vector as a quantity with direction and magnitude, is a quantity that converts a vector into another vector via a dot product. An example of this is given in equation (1.6): $\boldsymbol{L} = \boldsymbol{I} \cdot \boldsymbol{\Omega}$. The inertia tensor $\boldsymbol{I}$ gives us the angular momentum vector $\boldsymbol{L}$ when dotted with the rotation vector $\boldsymbol{\Omega}$. Note that the resultant vector $\boldsymbol{L}$ only points in the same direction as the applied vector $\boldsymbol{\Omega}$ in special cases that we discuss below.

Tensors in particular reference frames are represented by square matrices. Thus, the above relationship between $\boldsymbol{\Omega}$ and $\boldsymbol{L}$ can be expressed

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}. \tag{1.14}$$

Just as with vectors, there is a rule allowing us to obtain the matrix representation of a tensor in a new reference frame, given the matrix representation in the original reference frame. To see what this rule is, let us dot the angular momentum vector with the rotation matrix $\boldsymbol{U}$ that gives us the components of this vector in a new, primed reference frame:

$$\boldsymbol{L}' = \boldsymbol{U} \cdot \boldsymbol{L} = \boldsymbol{U} \cdot \boldsymbol{I} \cdot \boldsymbol{\Omega} = \boldsymbol{U} \cdot \boldsymbol{I} \cdot \boldsymbol{U}^T \cdot \boldsymbol{U} \cdot \boldsymbol{\Omega} = \boldsymbol{I}' \cdot \boldsymbol{\Omega}' \tag{1.15}$$

where the representations of the inertia tensor and the rotation vector in the primed reference frame are

$$\boldsymbol{I}' = \boldsymbol{U} \cdot \boldsymbol{I} \cdot \boldsymbol{U}^T \qquad \boldsymbol{\Omega}' = \boldsymbol{U} \cdot \boldsymbol{\Omega}. \tag{1.16}$$

Some words of clarification: First, from the above analysis we see that a tensor is not a matrix; it has a broader physical meaning independent of a given coordinate system. A tensor can be represented by a matrix in a particular coordinate system, but the matrix representation of the tensor in a different coordinate system is different, just as the representation of a vector in terms of a row or column matrix differs between different coordinate systems. Second, the rotation matrix $\boldsymbol{U}$ is not a tensor – it is just a matrix with a particular purpose; translate the representations of vectors and tensors from one coordinate system to another.

Examination of the definition of the inertia tensor in equation (1.7) shows that this tensor is *symmetric*; in other words, $I_{ij} = I_{ji}$ for all i and j. Another way to represent this condition is that $\boldsymbol{I}^T = \boldsymbol{I}$; the transpose of a symmetric matrix is equal to the original matrix. Let us ask the following question; is there a coordinate system in which the matrix representation of $\boldsymbol{I}$ is diagonal, i.e., $I_{ij} = 0$ for $i \neq j$? If there were such a coordinate system, then the dot product of a coordinate axis unit vector $\boldsymbol{e}$ with the tensor should yield $\boldsymbol{e}$ back again multiplied by a constant I (why?):

$$\boldsymbol{I} \cdot \boldsymbol{e} = \lambda \boldsymbol{e} = I \boldsymbol{J} \cdot \boldsymbol{e} \tag{1.17}$$

which we rewrite in the form

$$(\boldsymbol{I} - I\boldsymbol{J}) \cdot \boldsymbol{e} = 0. \tag{1.18}$$

The matrix representation of this in the original reference frame gives us a set of three homogeneous linear equations in the components of $\boldsymbol{e}$:

$$\begin{pmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = 0. \tag{1.19}$$

This set of equations has non-trivial solutions (components of $\boldsymbol{e}$ not equal to zero) only if the determinant of the $\boldsymbol{I} - I\boldsymbol{J}$ matrix equals zero. This determinant leads to a cubic polynomial in I. According to linear algebra, the roots of this polynomial are real. These roots are called the *eigenvalues* or characteristic values. For each of these roots there are values for the components of $\boldsymbol{e}$. As this is a linear, homogeneous system, if $\boldsymbol{e}$ is a solution, then any number times $\boldsymbol{e}$ is also a solution. Since we are seeking a unit vector, we resolve this ambiguity by normalizing $\boldsymbol{e}$ so that $|\boldsymbol{e}| = 1$.

If the three values of I, which we name I_1, I_2, and I_3, are distinct, then the corresponding unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are mutually orthogonal. The $\boldsymbol{e}_i$ are called *eigenvectors* or principal axes and they define the coordinate axes of a new coordinate system which we call the *principal axis* coordinate system. In this coordinate system the inertia tensor is diagonal, with the eigenvalues as the diagonal components:

$$\boldsymbol{I}_{PA} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{1.20}$$

The rotation matrix that carries vectors and tensors in the original coordinate system into the principal axis system is expressed in terms of the components of the eigenvectors:

$$\boldsymbol{U} = \begin{pmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ e_{3x} & e_{3y} & e_{3z} \end{pmatrix} \tag{1.21}$$

An important special case occurs when two or more of the eigenvectors are equal to each other. In this case, the $\boldsymbol{e}_i$ are not automatically orthogonal. However, in this case it is always possible to choose eigenvectors so that they are orthogonal. This choice is not unique, but any pair of orthogonal eigenvectors that lie in the plane defined by two non-orthogonal solutions to equation (1.19) associated with the non-unique eigenvalues are valid eigenvectors. If all three eigenvalues are equal, then any choice of three mutually orthogonal unit vectors constitutes a set of valid eigenvectors.

1.4 Forces and torques

If the force on a mass element dm of a solid is $\boldsymbol{f}dm$, then the total force on the solid is

$$\boldsymbol{F} = \int \boldsymbol{f} dm \tag{1.22}$$

and the total torque is

$$\boldsymbol{\tau} = \int \boldsymbol{r} \times \boldsymbol{f} dm. \tag{1.23}$$

If the reference system in which calculations are being done is accelerated with acceleration $\boldsymbol{A}$, then the force per unit mass has a physical part $\boldsymbol{f}_p$ and an inertial part $-\boldsymbol{A}$. In this case the total force takes the form

$$\boldsymbol{F} = \int \boldsymbol{f}_p dm - M\boldsymbol{A} \tag{1.24}$$

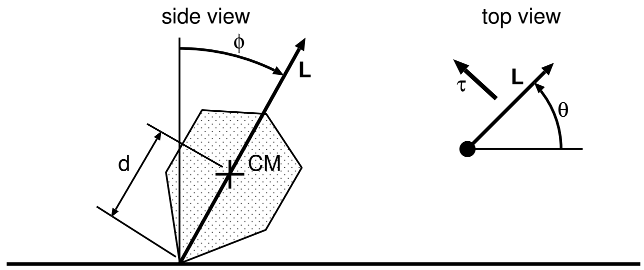

Figure 1.2: Sketch of a spinning top under the influence of gravity.

where M is the total mass of the solid body, and the torque is

$$\boldsymbol{\tau} = \int \boldsymbol{r} \times \boldsymbol{f}_p dm - \int \boldsymbol{r} dm \times \boldsymbol{A} = \int \boldsymbol{r} \times \boldsymbol{f}_p dm - \boldsymbol{R}_{CM} \times \boldsymbol{A} \tag{1.25}$$

where the definition of center of mass has been used. In the special case in which the origin of the coordinate system is at the center of mass, then $\boldsymbol{R}_{CM} = 0$ by definition, and the inertial force produces no torque. The same is true for any other force that has a constant value per unit mass, such as that from a constant gravitational field.

The necessary conditions for static equilibrium of a solid body are simply

$$\boldsymbol{F} = 0 \qquad \boldsymbol{\tau} = 0. \tag{1.26}$$

1.5 Elementary treatment of top

Figure 1.2 shows two views of a spinning top, with angle ϕ giving the tilt of the top from the vertical and θ giving the azimuthal organization in the horizontal plane. In this treatment, we assume that the top is spinning rapidly so that assuming that the angular momentum vector $\boldsymbol{L}$ is aligned with the axis of rotation is a good approximation. The angular momentum changes with time due to the torque (illustrated in the right panel) exerted by the gravitational and the surface forces on the top. The top is assumed to be free to slide without friction on the horizontal surface, but is subject to an upward normal force equal in magnitude to the downward gravitational force which acts at the center of mass of the top. The magnitude of the torque due to these two forces is

$$\tau = mgd \sin \phi \tag{1.27}$$

where m is the mass of the top and g is the gravitational field strength, so that in time dt the change in the angular momentum is

$$dL = \tau dt = mgd \sin \phi dt. \tag{1.28}$$

The change in angular momentum is in the horizontal plane and normal to the angular momentum vector in the sense of increasing θ with time. The change in θ in time dt is

$$d\theta = \frac{dL}{L \sin \phi} = \frac{mgd \sin \phi dt}{L \sin \phi} \tag{1.29}$$

so that the precession frequency of the top is

$$\Omega \equiv \frac{d\theta}{dt} = \frac{mgd}{L}. \tag{1.30}$$

The angular momentum can be expressed in terms of the moment of inertia $I = \boldsymbol{n} \cdot \boldsymbol{I} \cdot \boldsymbol{n}$ of the top ($\boldsymbol{n}$ is the unit vector aligned with the axis of rotation) and the angular rotation rate of the top ω, $L = I\omega$, which means that

$$\Omega = \frac{mgd}{I\omega}. \tag{1.31}$$

Thus, the precessional frequency of the top is independent of the tilt angle ϕ and increases as the spin rate of the top decreases.

1.6 Problems

1. Show that the motion of a solid body may be described by a single rotation (without an accompanying translation) if some point in the solid body remains stationary in this motion.

2. Show that an eigenvector of a symmetric tensor yields the eigenvector back again times a scalar when dotted with the tensor. Hint: Work in the principal axis coordinate system. If the result is valid in this system, it is valid in all coordinate systems.

3. Prove that the eigenvalues of a symmetric tensor are real.

4. Prove that any two eigenvectors of a symmetric tensor are orthogonal when the associated eigenvalues are not equal.

5. Asymmetric rotator:

   (a) Compute the inertia tensor for the asymmetric rotator in figure 1.3.

   (b) Find the eigenvalues and normalized eigenvectors of this tensor.

   (c) Show that the eigenvalues are orthogonal.

   (d) Find the orthogonal rotation matrix $\boldsymbol{U}$ that transforms vectors and tensors into the principal axis reference frame.

   (e) Calculate $\boldsymbol{U} \cdot \boldsymbol{I} \cdot \boldsymbol{U}^T$ and verify that the result is diagonal with the eigenvalues on the diagonal.

   (f) What happens to the eigenvalues and eigenvectors if $w = d$? In particular, is there an alternate set of eigenvectors?

6. If the rotation vector lies along the x axis for the rotator of problem 5 and is given by $\boldsymbol{\Omega} = (\Omega, 0, 0)$, find the angular momentum vector and the kinetic energy when the rotator is oriented as in figure 1.3. You will need the results of problem 5.

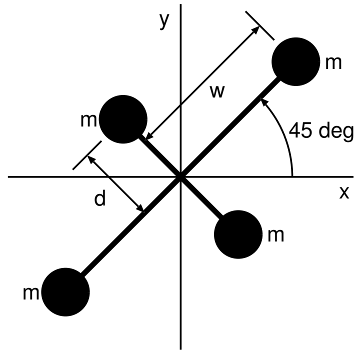

Figure 1.3: Configuration of 4 masses connected by very light rods.

7. Laterally constrained top:

    (a) Recalculate the precessional frequency of the top derived in section 1.5 on the assumption that the bottom point of the top is constrained to remain in one place. Hint: In this case the center of mass executes a circular motion due to the precession of the top. In the reference frame of the center of mass of the top, there is therefore an outward centrifugal force acting at the center of mass, balanced by a corresponding centripetal force acting at the base of the top. This force pair exerts an additional torque on the top.

    (b) From the above results, determine the minimum angular momentum needed to keep the top from falling over.