Chapter 1

Rotation

In this chapter we first relate the time rate of change of an arbitrary vector in a rotating reference frame to its time tendency in an inertial frame. We then use this to derive the inertial forces arising in a rotating reference frame. We also use this analysis to derive the Euler equations for a rotating body. Applications are then presented.

1.1 Rotating reference frames

Generalizing the analysis at the beginning of the chapter on solid bodies, which considered the time rate of change of the position of a point fixed in a rotating body, we recognize that this analysis extends to any vector $\boldsymbol{e}$ fixed in a solid body,

$$\frac{d\boldsymbol{e}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{e}, \tag{1.1}$$

where $\boldsymbol{\Omega}$ is the rotation vector, as illustrated in figure 1.1. Let us now imagine a Cartesian coordinate system embedded in a solid body with unit vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. An arbitrary vector $\boldsymbol{A}$ can be represented in terms of its components in this reference frame:

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}. \tag{1.2}$$

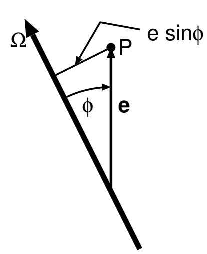

Figure 1.1: Illustration for obtaining the time rate of change of a vector in a rotating reference frame.

Taking the time derivative of this with respect to an absolute, inertial reference frame yields

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{abs} = \frac{dA_x}{dt}\boldsymbol{i} + \frac{dA_y}{dt}\boldsymbol{j} + \frac{dA_z}{dt}\boldsymbol{k} = A_x\frac{d\boldsymbol{i}}{dt} + A_y\frac{d\boldsymbol{j}}{dt} + A_z\frac{d\boldsymbol{k}}{dt}. \tag{1.3}$$

The first three components of this expansion constitute the time derivative of $\boldsymbol{A}$ with respect to the moving reference frame, $(d\boldsymbol{A}/dt)_{rel}$. Using equation (1.1), the second three terms can be written

$$A_x\frac{d\boldsymbol{i}}{dt} + A_y\frac{d\boldsymbol{j}}{dt} + A_z\frac{d\boldsymbol{k}}{dt} = A_x\boldsymbol{\Omega}\times\boldsymbol{i} + A_y\boldsymbol{\Omega}\times\boldsymbol{j} + A_z\boldsymbol{\Omega}\times\boldsymbol{k} = \boldsymbol{\Omega}\times\boldsymbol{A} \tag{1.4}$$

so that equation (1.3) becomes

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{abs} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{rel} + \boldsymbol{\Omega}\times\boldsymbol{A}. \tag{1.5}$$

Since $\boldsymbol{v} = d\boldsymbol{r}/dt$, equation (1.5) tells us that

$$\boldsymbol{v}_{abs} = \boldsymbol{v}_{rel} + \boldsymbol{\Omega}\times\boldsymbol{r} \tag{1.6}$$

where $\boldsymbol{v}_{abs}$ and $\boldsymbol{v}_{rel}$ are respectively the velocities in the inertial and rotating frames. The inertial frame time derivative of this equation gives us the acceleration in the inertial frame

$$\begin{aligned}
\boldsymbol{a}_{abs} &= \left(\frac{d\boldsymbol{v}_{abs}}{dt}\right)_{abs} \\
&= \left(\frac{d\boldsymbol{v}_{abs}}{dt}\right)_{rel} + \boldsymbol{\Omega}\times\boldsymbol{v}_{abs} \\
&= \left(\frac{d\boldsymbol{v}_{rel}}{dt}\right)_{rel} + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r} + \boldsymbol{\Omega}\times\boldsymbol{v}_{rel} + \boldsymbol{\Omega}\times(\boldsymbol{v}_{rel} + \boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \boldsymbol{a}_{rel} + 2\boldsymbol{\Omega}\times\boldsymbol{v}_{rel} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r}.
\end{aligned} \tag{1.7}$$

Line 2 of equation (1.7) comes from applying equation (1.5) with $\boldsymbol{A} = \boldsymbol{v}_{abs}$ while in line 3 $\boldsymbol{v}_{abs}$ is eliminated in favor of $\boldsymbol{v}_{rel}$ using equation (1.6). Finally, in line 4 the results are rearranged and $(d\boldsymbol{v}_{rel}/dt)_{rel}$ is recognized as the acceleration in the rotating frame $\boldsymbol{a}_{rel}$. The time derivative of $\boldsymbol{\Omega}$ in the inertial and rotating frames is the same since by definition $\boldsymbol{\Omega}$ lies along the rotation axis.

Let us now consider Newton's second law applied to a mass m and take terms 2-4 on the right side of the last line of equation (1.7) to the left side of this equation, where they become inertial forces after multiplying through by m:

$$\boldsymbol{F} + \boldsymbol{F}^* = m\boldsymbol{a}_{rel} \tag{1.8}$$

where

$$\boldsymbol{F}^* = \boldsymbol{F}_{cor} + \boldsymbol{F}_{cen} + \boldsymbol{F}_{accel} = -2m\boldsymbol{\Omega}\times\boldsymbol{v}_{rel} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) - m\dot{\boldsymbol{\Omega}}\times\boldsymbol{r} \tag{1.9}$$

is the sum of three inertial forces. The quantity $\boldsymbol{F}_{cor} = -2m\boldsymbol{\Omega} \times \boldsymbol{v}_{rel}$ is called the Coriolis force, $\boldsymbol{F}_{cen} = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ is the centrifugal force, and $\boldsymbol{F}_{accel} = -m\dot{\boldsymbol{\Omega}} \times \boldsymbol{r}$ is the inertial force associated with changes in time of the rotation vector $\boldsymbol{\Omega}$. Notice that the Coriolis force is normal to both the rotation axis and the velocity of the mass in the rotating coordinate system. The centrifugal force points away from the rotation axis and has the magnitude $\Omega^2 d$ where d is the distance of the mass from the rotation axis. The quantity $\dot{\boldsymbol{\Omega}}$ can represent either changes in magnitude or direction or both.

1.2 Euler equations

It is useful to represent the torque equation in a rotating reference frame. Taking advantage of equation (1.5), we find that

$$\boldsymbol{\tau} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{abs} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{rel} + \boldsymbol{\Omega} \times \boldsymbol{L}. \tag{1.10}$$

Taking the rotating reference frame to be a reference frame fixed to a solid body is an interesting choice, since the inertia tensor doesn't change with time in this frame. To further simplify things, we choose as our reference frame the principal axes of the inertia tensor, which allows us to write

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}. \tag{1.11}$$

Substituting equation (1.11) into equation and omitting the torque (which is hard to specify in the rotating frame of the object) (1.10) results in the Euler equations:

$$\begin{aligned} I_1 \frac{d\Omega_1}{dt} &= (I_2 - I_3)\Omega_2\Omega_3 \\ I_2 \frac{d\Omega_2}{dt} &= (I_3 - I_1)\Omega_3\Omega_1 \\ I_3 \frac{d\Omega_3}{dt} &= (I_1 - I_2)\Omega_1\Omega_2. \end{aligned} \tag{1.12}$$

These equations show how the rotation vector moves relative to the principal axis coordinate system embedded in the solid body. However, it gives us no hint as to how these axes are aligned relative to the original inertial reference frame.

1.2.1 Axially symmetric body with no torque

Suppose that the solid body exhibits symmetry of rotation about axis 3. In this case $I_1 = I_2$ and the third equation (1.12) becomes

$$\frac{d\Omega_3}{dt} = 0 \tag{1.13}$$

in the case of no torque. In this case the first two equations become

$$\begin{aligned} \frac{d\Omega_1}{dt} &= -\Omega_N\Omega_2 \\ \frac{d\Omega_2}{dt} &= \Omega_N\Omega_1 \end{aligned} \tag{1.14}$$

where

$$\Omega_N = \frac{(I_3 - I_1)\Omega_3}{I_1} \tag{1.15}$$

is a constant called the *precession* frequency.

This equation is easy to solve using complex methods by defining $\Omega = \Omega_1 + i\Omega_2$ and rewriting equation (1.14) in complex form:

$$\frac{d\Omega}{dt} = i\Omega_N\Omega. \tag{1.16}$$

The solution to this equation is

$$\Omega = \Omega_0 \exp(i\Omega_N t) \tag{1.17}$$

where $\Omega_0 = |\Omega| = (\Omega_1^2 + \Omega_2^2)^{1/2}$ is a constant. Physically, the rotation vector traces out a cone as it precesses about principal axis 3 in the reference frame of the rotating solid. If Ω_N is positive, the movement is the same as that of the rotation about axis 3, in the opposite direction if Ω_N is negative.

1.2.2 Asymmetric body – stability analysis

Suppose that an asymmetric body (no degeneracy in the eigenvalues of the inertia tensor) with no torque acting has the rotation vector parallel to a particular axis in the principal axis reference frame, say, axis 3, so that $\Omega_1 = \Omega_2 = 0$. The Euler equations with zero torque reduce to

$$\frac{d\Omega_1}{dt} = \frac{d\Omega_2}{dt} = \frac{d\Omega_3}{dt} = 0 \tag{1.18}$$

in this case, i.e., rotation around a principal axis is an equilibrium state.

The question is whether this rotation represents a stable or an unstable equilibrium. Linearizing the Euler equations and solving yields an answer to this question. Letting $\Omega_3 = \Omega_0 = \mathit{constant}$ and taking Ω_1 and Ω_2 as small quantities yields equations linear in Ω_1 and Ω_2. Their solution reveals that If I_3 is the maximum or minimum eigenvalue of the inertia tensor, then the rotation is stable. However, if I_3 is the middle value, then the motion is unstable.

1.3 Symmetric top using Lagrangian method

We now make a more sophisticated analysis of a spinning top under the influence of gravity. To do this we need to obtain a transformation of components of the rotation vector from

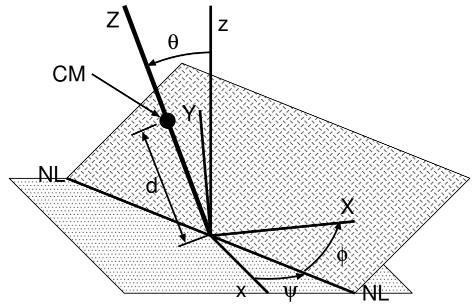

Figure 1.2: Definition sketch of the coordinate axes (x, y, z) in the inertial frame and the principal axis coordinates (X, Y, Z) of the spinning top as well as the Euler angles (θ, ϕ, ψ) that connect them. CM denotes the center of mass of the top and NL the line of nodes.

Cartesian coordinates in the inertial reference frame to their components $(\Omega_1, \Omega_2, \Omega_3) \equiv (\Omega_X, \Omega_Y, \Omega_Z)$ in the principal axis coordinate system of the top. Figure 1.2 shows the Euler angles that connect these two reference frames. Staring very hard at figure 1.2 indicates that

$$\begin{aligned} \Omega_X =& \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta \\ \Omega_Y =& -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\sin\theta \\ \Omega_Z =& \dot{\phi} + \dot{\psi}\cos\theta. \end{aligned} \tag{1.19}$$

The kinetic energy of the top is

$$\begin{aligned} T =& \frac{1}{2}\left[(\Omega_X^2 + \Omega_Y^2)I_1 + \Omega_Z^2 I_3\right] \\ =& \frac{1}{2}\left[I_1\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + I_3\left(\dot{\phi} + \dot{\psi}\cos\theta\right)^2\right] \end{aligned} \tag{1.20}$$

where I_3 is the moment of inertia for rotation about the top axis Z and $I_1 = I_2$ are the moments of inertia for rotations about the X and Y axes in figure 1.2. The potential energy due to gravity is

$$V = mgd\cos\theta, \tag{1.21}$$

leading to the Lagrangian

$$L = \frac{1}{2}\left[I_1\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + I_3\left(\dot{\phi} + \dot{\psi}\cos\theta\right)^2\right] - mgd\cos\theta. \tag{1.22}$$

The three resulting equations of motion are

$$I_1\frac{d^2\theta}{dt^2} - I_1\dot{\psi}^2\sin\theta\cos\theta + I_3\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\dot{\psi}\sin\theta - mgd\sin\theta = 0 \tag{1.23}$$

$$\frac{d}{dt}\left[I_3\left(\dot{\phi} + \dot{\psi}\cos\phi\right)\right] \equiv \frac{dp_\phi}{dt} = 0 \tag{1.24}$$

and

$$\frac{d}{dt}\left[I_1\dot{\psi}\sin^2\theta + I_3\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\cos\theta\right] \equiv \frac{dp_\psi}{dt} = 0, \tag{1.25}$$

from which it is clear that there are two conserved generalized momenta

$$p_\phi = I_3\left(\dot\phi + \dot\psi\cos\phi\right) \tag{1.26}$$

and

$$p_\psi = I_1\dot\psi\sin^2\theta + p_\phi\cos\theta. \tag{1.27}$$

The first of these p_ϕ is the component of angular momentum about the axis of symmetry of the top Z and the second p_ψ is the component of angular momentum about the vertical axis z, which includes both the vertical component of the rotation of the top and its precession.

Solving equation (1.27) for $\dot\psi$

$$\dot\psi = \frac{p_\psi - p_\phi\cos\theta}{I_1\sin^2\theta} \tag{1.28}$$

allows us to eliminate $\dot\psi$ and $\dot\phi$ from equation (1.25), resulting in

$$I_1\frac{d^2\theta}{dt^2} + \frac{(p_\psi - p_\phi\cos\theta)(p_\phi - p_\psi\cos\theta)}{I_1\sin^3\theta} - mgd\sin\theta = 0. \tag{1.29}$$

This may be rearranged in the following form

$$\frac{d^2\theta}{dt^2} - \frac{(p_\psi - p_\phi\cos\theta)(p_\psi - p_\phi/\cos\theta) + D\sin^4\theta/\cos\theta}{I_1^2(\sin^3\theta/\cos\theta)} = 0 \tag{1.30}$$

where $D = mgdI_1$. Written this way, the numerator of the second term of equation (1.30) is a second order polynomial in p_ψ, which can be factored, with roots

$$p_\psi^\pm = \frac{p_\phi\left(1+\cos^2\theta\right) \pm \left(p_\phi^2 - 4D\cos\theta\right)^{1/2}\sin^2\theta}{2\cos\theta}. \tag{1.31}$$

Given these roots, the governing equation for θ can finally be rewritten

$$\frac{d^2\theta}{dt^2} - \frac{(p_\psi - p_\psi^+)(p_\psi - p_\psi^-)}{I_1^2(\sin^3\theta/\cos\theta)} = 0. \tag{1.32}$$

This factorization demonstrates that two equilibrium states of the top exist, one with $p_\psi^+ = p_\psi$, the other with $p_\psi^- = p_\psi$, with corresponding equilibrium values of θ, which we call θ^+ and θ^-.

In the limit of zero gravity ($D = 0$), we are back to the case of a symmetric body with no torques. In this case equation (1.31) tells us that

$$p_\psi^+ = p_\phi/\cos\theta^+ \qquad p_\phi^- = p_\phi\cos\theta^-. \tag{1.33}$$

Referring to equation (1.28), we note that the corresponding equilibrium values of $\dot\psi$ are

$$\dot\psi^+ = \frac{p_\phi}{I_1\cos\theta^+} \qquad \dot\psi^- = 0. \tag{1.34}$$

The "minus" equilbrium case is rather peculiar; since $\dot{\theta}$ is zero in equilibrium, equation (1.19) tells us that $\mathbf{\Omega} = (0, 0, \dot{\phi})$, i.e., the top is spinning serenely about its 3rd principal axis, which is tilted an angle θ^- from the vertical, a completely steady, stable configuration in the absence of gravity. On the other hand, the "plus" equilibrium has $(\Omega_x^2 + \Omega_y^2)^{1/2} = \Omega_z \tan\theta_0$ according to equation (1.19). The rotation about the vertical axis of this equilibrium state just represents the nutational motions studied in section 1.2.1, but viewed from an inertial reference frame.

Examination of equation (1.31) reveals that these two equilibrium states converge as gravity gets stronger relative to the spin angular momentum p_ϕ, culminating in the limit in which

$$p_{\phi crit} = 2\left(D\cos\theta^+\right)^{1/2} = 2\left(D\cos\theta^-\right)^{1/2}. \tag{1.35}$$

Equation (1.31) becomes non-physical for $p_\phi < p_{\phi crit}$; physically, as the spin rate of a top decreases due to friction, this represents the point at which it can no longer precess, thus falling over on its side. Equation (1.28) with p_ψ^- and θ_0 substituted gives the exact precession rate of a top:

$$\dot{\psi}^- = \frac{p_\phi - (p_\phi^2 - 4D\cos\theta^-)^{1/2}}{2I_1\cos\theta^-}. \tag{1.36}$$

1.4 Rapidly rotating top

If $p_\phi \gg p_{\phi crit}$, then we have the scenario of a rapidly rotating top relative to the minimum rotation rate needed to keep it from falling over. In this case equation (1.31) may be approximated as follows:

$$p_\psi^+ = \frac{p_\phi}{\cos\theta} - \frac{D\sin^2\theta}{p_\phi} \tag{1.37}$$

$$p_\psi^- = p_\phi\cos\theta + \frac{D\sin^2\theta}{p_\phi}. \tag{1.38}$$

This approximation facilitates the linearization of equation (1.32).

Linearization can be performed about either equilibrium state. Here we choose to linearize about the $p_\psi^- = p_\psi$ state. In this equilibrium state $\theta = \theta^-$. Substituting $\theta = \theta^-$ and $p_\psi^- = p_\psi$ into equation (1.38) yields an implicit equation for θ^- in terms of p_ψ:

$$p_\psi = p_\phi\cos\theta^- + \frac{D\sin^2\theta^-}{p_\phi}. \tag{1.39}$$

Fortunately, we do not actually have to solve this equation for θ^-. We simply need to linearize equation (1.38) about this state. Letting $\theta = \theta^- + \theta'$, we find that

$$p_\psi^- \approx p_\psi + \left(-p_\phi\sin\theta^- + \frac{2D\sin\theta^-\cos\theta^-}{p_\phi}\right)\theta', \tag{1.40}$$

which means that

$$p_\psi - p_\psi^- = \left(p_\phi\sin\theta^- - \frac{2D\sin\theta^-\cos\theta^-}{p_\phi}\right)\theta' \tag{1.41}$$

is first order in θ'. In principle, everything else in the second term of equation (1.32) should be expanded in powers of θ', but this would be a pointless exercise, since the terms beyond the zeroth term would yield θ' to second and higher powers in the second term as a whole. Since these terms are dropped in the linearization, we can simply substitute $\theta \to \theta^-$ and we are done.

Using

$$\left(p_\psi^- - p_\psi^+\right)_{\theta^-} = -\frac{\sin^2\theta^-}{\cos\theta^-}\left(p_\phi - \frac{2D\cos\theta^-}{p_\phi}\right), \tag{1.42}$$

equation (1.32) becomes

$$\frac{d^2\theta'}{dt^2} + \frac{1}{I_1^2}\left(p_\phi - \frac{2D\cos\theta^-}{p_\phi}\right)^2 \theta' = 0. \tag{1.43}$$

This is just a harmonic oscillator equation for small oscillations of the top in θ about its equilibrium value with oscillation frequency

$$\omega = \frac{1}{I_1}\left(p_\phi - \frac{2D\cos\theta^-}{p_\phi}\right). \tag{1.44}$$

Physically, these oscillations correspond to the bobbing motions of a top as it precesses about the vertical axis. This type of motion is called *nutation*. Variations in θ also cause variations in the precessional frequency $\dot{\psi}$, so the top traces out looping type patterns as it nutates.

1.5 Problems

1. Compute the magnitude and direction of the horizontal part of the Coriolis force due to the earth's rotation for an object of mass m moving horizontally on the earth at latitude ϕ.

2. Using the above result, compute the motion of the bob of a pendulum as a function of time at latitude ϕ. Investigate both simple back and forth oscillations and the small amplitude circular oscillations of a spherical pendulum. Hint: Ignore the vertical component of the Coriolis force, as it is negligible compared to gravity. Also, assume the small angle approximation in the movement of the pendulum bob and further assume that the motion is essentially horizontal.

3. Show that the centrifugal force is conservative and therefore may be represented by a potential energy. Compute the potential energy distribution in a reference frame rotating about a vertical axis passing through the origin at rate Ω.

4. The geopotential field of the earth is the net potential energy per unit mass due to a combination of gravity and the centrifugal force. The value of g is minus the gradient of the geopotential.

(a) Assuming a spherical earth, compute the difference in g between the equator and the poles. (In reality, the shape of the earth adjusts to fit constant values of geopotential, so the actual result is different.)

(b) Calculate the distance from the center of the earth at which g goes to zero over the equator.

5. Carry out the stability analysis for for the asymmetric rotator described in section 1.2.2.

6. Show in the limits that $p_\phi^2 \gg 4D\cos\theta_0$ and $\dot{\phi} \gg \dot{\psi}$ (in the definition of p_ϕ) that the precession rate of the top given by equation (1.36) reduces to the approximate value deduced in the previous chapter.

7. Carry out a stability analysis of the rapidly spinning top relative to the $p_\psi = p_\psi^+$ equilibrium state in analogy to that for the $p_\psi = p_\psi^-$ case given in section 1.4.