# Chapter 1

# Relativistic kinematics

## 1.1 Spacetime Pythagorean theorem

We first review what we know about the spacetime Pythagorean theorem. Assuming for simplicity that the speed of light $c = 1$, then referring to the triangles in figure 1.1.1, we know that

$$x^2 - t^2 = I^2$$

for a spacelike hypotenuse and

$$t^2 - x^2 = \tau^2$$

for a timelike hypotenuse. The quantity $I$ is the spacetime interval and $\tau$ is the proper time. They are clearly related by $I^2 = -\tau^2$, so defining both is just a convenience so that the spacelike and timelike cases can be considered separately.

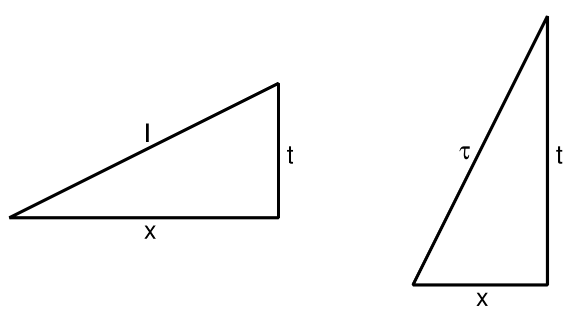


Figure 1.1.1: Triangles for spacetime Pythagorean theorem.

The Pythagorean theorem in ordinary space is just

$$r^2 = x^2 + y^2$$

where $r$ is the hypotenuse. Note that we can turn this into the spacetime Pythagorean theorem by setting $y = it$, where $i = (-1)^{1/2}$, which results in $y^2 = -t^2$. Don't try to interpret this physically, it is just a mathematical trick, albeit a useful one, as we shall see!

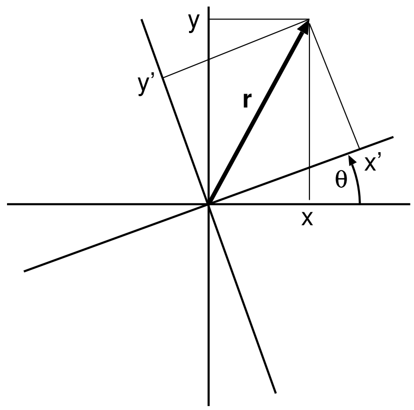


Figure 1.2.1: Illustration of vector $\boldsymbol{r}$ resolved into components in two reference frames.

## 1.2 Rotations in two space dimensions

Changing reference systems in spacetime is somewhat like transforming to a rotated coodinate system in ordinary space. Let's first review the latter in order to get hints as to how to do the former in a systematic way.

Suppose we have a position vector $\boldsymbol{r}$ with components $(x, y)$ in the unrotated frame and $(x', y')$ in a frame rotated by an angle $\theta$ in the counterclockwise direction, as shown in figure 1.2.1. This vector can be resolved into components in the primed and unprimed reference frame:

$$\boldsymbol{r} = x'\hat{\boldsymbol{i}}' + y'\hat{\boldsymbol{j}}' = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}. \tag{1.2.1}$$

Dotting with $\hat{\boldsymbol{i}}'$ and $\hat{\boldsymbol{j}}'$results in two scalar equations

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \tag{1.2.2}$$

that tell us how to get $(x', y')$ from $(x, y)$ and the rotation angle $\theta$. It is easy to show that $\hat{\boldsymbol{i}}' \cdot \hat{\boldsymbol{i}} = \cos\theta$, $\hat{\boldsymbol{i}}' \cdot \hat{\boldsymbol{j}} = \sin\theta$, etc.

## 1.3 Lorentz transformation

Let's now use the insight that spacetime is equivalent to a Euclidean space in which one component (the time component) is imaginary. Setting $y = it$, the above equations become

$$\begin{aligned} x' &= x\cos\theta + t(i\sin\theta) \\ t' &= x(i\sin\theta) + t\cos\theta. \end{aligned} \tag{1.3.1}$$

(Note the change in sign of the first term in the second equation.)

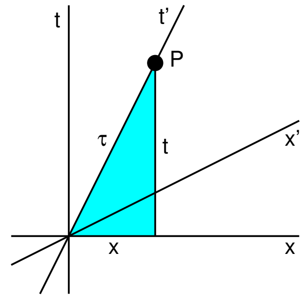


Figure 1.3.1: Test triangle in spacetime.

The only problem is that $(x, t)$ and the primed counterparts are real, which means that both $\cos\theta$ and $i\sin\theta$ must be real also. Let's write the sine and cosine in terms of exponentials using Euler's theorem and see what this reality condition does to $\theta$:

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2} \qquad i\sin\theta = \frac{\exp(i\theta) - \exp(-i\theta)}{2}. \tag{1.3.2}$$

These terms may be made real by making $\theta$ imaginary. Setting $\theta = i\phi$, where $\phi$ is real, results in

$$\cos\theta = \frac{\exp(\phi) + \exp(-\phi)}{2} \equiv \cosh\phi \tag{1.3.3}$$

and

$$i\sin\theta = -\frac{\exp(\phi) - \exp(-\phi)}{2} \equiv -\sinh\phi. \tag{1.3.4}$$

Substituting these expressions results in

$$\begin{aligned} x' &= x\cosh\phi - t\sinh\phi \\ t' &= -x\sinh\phi + t\cosh\phi. \end{aligned} \tag{1.3.5}$$

Things are weird in relativity as usual; a change in velocity reference frame is equivalent to a rotation through an imaginary angle!

Figure 1.3.1 illustrates a test point P, which has spacetime coordinates $(x, t)$ in the unprimed coordinate system and the coordinates $(0, \tau)$ in the primed system – the $x$ coordinate in the primed frame is zero because P lies on the primed time axis. The slope of a world line parallel to the $t'$ axis is

$$slope = \frac{t}{x} = \frac{1}{\beta} \tag{1.3.6}$$

where $\beta = v/c = v$ is the non-dimensional velocity of the object represented by the world line. Since the point P is on the $t'$ axis, $x' = 0$, which from the first line of equation (1.3.5) tells us that

$$\beta = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi. \tag{1.3.7}$$

To make further progress, we need the identity

$$\cosh^2\phi - \sinh^2\phi = 1. \tag{1.3.8}$$

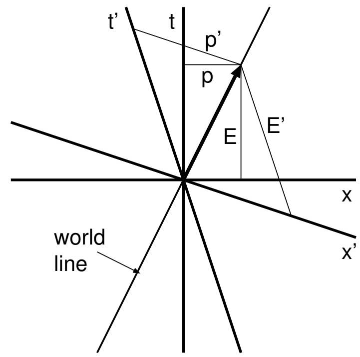


Figure 1.4.1: Definition sketch for the addition of velocities in relativity.

Using equation (1.3.7), we easily find that $\sinh\theta = \beta\cosh\theta$ and with the above identity we get

$$\cosh\phi = \frac{1}{(1-\beta^2)^{1/2}} \equiv \gamma \tag{1.3.9}$$

and

$$\sinh\phi = \beta\gamma. \tag{1.3.10}$$

From these and equations (1.3.5) we find what is called the Lorentz transformation:

$$\begin{aligned} x' &= \gamma x - \beta\gamma t \\ t' &= -\beta\gamma x + \gamma t. \end{aligned} \tag{1.3.11}$$

We have derived the Lorentz transformation for the space and time components of a position 4-vector. However, the derivation is equally valid for any 4-vector, such as a displacement in spacetime, a wave 4-vector, or the energy-momentum 4-vector.

## 1.4 Addition of velocities

The Lorentz transformations make it easy to derive the relativistic velocity addition formula. Referring to figure 1.4.1, we imagine an object (like a space ship) moving to the right with (non-dimensional) velocity $v$ relative to the unprimed reference frame. The energy-momentum 4-vector $(p, E)$ is parallel to the world line, which means that

$$v = \frac{p}{E}. \tag{1.4.1}$$

The primed frame is moving to the left with speed $\beta$, which means that its velocity is $-\beta$. The components of the energy-momentum vector in the primed frame $(p', E')$ are given by the Lorentz transformations, where we replace $(x, t)$ by $(p, E)$:

$$\begin{aligned} p' =& \gamma p + \beta\gamma E \\ E' =& \beta\gamma p + \gamma E. \end{aligned} \tag{1.4.2}$$

Realizing that the velocity of the spaceship in the primed frame is $v' = p'/E'$, we see that

$$v' = \frac{p'}{E'} = \frac{p + \beta E}{\beta p + E} = \frac{v + \beta}{1 + \beta v} \tag{1.4.3}$$

where we have divided the numerator and denominator by $E$ in the last step. Equation (1.4.3) is just the velocity addition formula.

## 1.5 Problems

1. Explain where the minus sign comes from in the second line of equation (1.2.2).
2. Prove the identity given in equation (1.3.8). Hint: Write the cosh and sinh in terms of exponentials.
3. Invert the Lorentz transformation to get $(x, t)$ in terms of $(x', t')$.
4. Use the Lorentz transformation to compute $\tau$ in terms of $t$ and $\beta$ in figure 1.3.1.
5. Use the Lorentz transformation to derive the Lorentz contraction.
6. A particle of mass $m$ at rest has energy-momentum 4-vector $(0, m)$ (recall that we are setting $c = 1$). Use the Lorentz transformation to find its energy and momentum moving to the left with velocity $-\beta$ ($\beta > 0$).
7. How do you think the Lorentz transformation generalizes to 3 space dimensions assuming that the velocity is still in the $x$ direction?