Chapter 1

Relativistic dynamics

1.1 Principle of relativity

We are familiar with the idea that physical laws must be valid in all inertial reference frames; this is called the principle of relativity. Mathematically, this is guaranteed in different ways for different types of equations. A law written in terms of relativistic scalars

$$scalar = scalar, \tag{1.1.1}$$

satisfies this condition, since scalars are the same in all reference frames. If the law is written in terms of 4-vectors, then we have a *covariant* relationship

$$\textit{4-vector} = \textit{4-vector}, \tag{1.1.2}$$

which means that the left and right sides vary in the same way with changes of coordinate system. This also satisfies the principle of relativity.

An example of the first is the definition of mass m in terms of the energy-momentum 4-vector $\underline{p} = (\boldsymbol{p}, E)$:

$$m^2 = -\underline{p} \cdot \underline{p} = E^2 - p^2. \tag{1.1.3}$$

Both sides of this equation are scalars; the mass m by definition and the dot product of the 4-momentum with itself via the properties of the dot product. The relationship between the wave 4-vector of a matter wave and the 4-momentum is an example of the second type:

$$\underline{p} = \hbar \underline{k}. \tag{1.1.4}$$

The components in different reference frames are different, but the components on the two sides change in similar ways since both sides of the equation are 4-vectors.

1.2 Relativistic form of Newton's second law

The relativistic form of Newton's second law is

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}. \tag{1.2.1}$$

This is correct, but it is not expressed in covariant form because (1) it is a relationship between space vectors only and (2) the dt is the timelike component of a displacement 4-vector and not a scalar. This makes the conventional definition of force highly inconvenient in the relativistic context.

If the object in question is moving at speed v, such that $\gamma = (1 - v^2)^{-1/2}$ (again, we set the speed of light $c = 1$), then we can rectify the second problem by eliminating the increment in time dt in favor of the increment in proper time $d\tau$ along the trajectory of the object, $dt = \gamma d\tau$, so that Newton's second law becomes

$$\boldsymbol{M} = \gamma \boldsymbol{F} = \frac{d\boldsymbol{p}}{d\tau} \tag{1.2.2}$$

where $\boldsymbol{M} = \gamma \boldsymbol{F}$ is called the *Minkowski force.* Notice that the Minkowski force and $d\tau$ differ respectively from the ordinary force $\boldsymbol{F}$ and the ordinary time increment dt when the object is moving at relativistic speeds.

This completes the first step in making Newton's second law covariant. The second step is inferring the timelike component of this equation. This is fairly obvious, as $\boldsymbol{p}$ is the spacelike component of the 4-momentum vector, the timelike component of which is the energy E:

$$M_t = \frac{dE}{d\tau} \tag{1.2.3}$$

where M_t is the timelike component of the Minkowski 4-force. The above equation indicates that M_t is just the proper time rate of change of energy, or something like the power. The complete, covariant form of Newton's second law is thus

$$\underline{M} = \frac{d\underline{p}}{d\tau} \tag{1.2.4}$$

where $\underline{M} = (\boldsymbol{M}, M_t)$.

Given some force $\boldsymbol{F}$, it is clear how to compute the spacelike part of the Minkowski force; just set $\boldsymbol{M} = \gamma \boldsymbol{F}$. However, equation (1.2.3) is not a particularly convenient way to compute the timelike part M_t. An easier way comes from dotting equation (1.2.4) with the 4-momentum $\underline{p}$:

$$\underline{p} \cdot \underline{M} = \underline{p} \cdot \frac{d\underline{p}}{d\tau} = \frac{1}{2}\frac{d(\underline{p} \cdot \underline{p})}{d\tau} = -\frac{1}{2}\frac{dm^2}{d\tau} = 0 \tag{1.2.5}$$

where we have used the fact that the length squared of $\underline{p}$ is minus the mass m squared. Since the mass of the object is a constant scalar (no virtual masses here!), it doesn't change with time, resulting in

$$\underline{p} \cdot \underline{M} = \boldsymbol{p} \cdot \boldsymbol{M} - E M_t = 0. \tag{1.2.6}$$

Since $\boldsymbol{p}/E = \boldsymbol{v}$, we find that

$$M_t = \boldsymbol{v} \cdot \boldsymbol{M} = \gamma \boldsymbol{v} \cdot \boldsymbol{F}. \tag{1.2.7}$$

In words, M_t is just γ times the power, or work done per unit time (not proper time) by the force $\boldsymbol{F}$ on the object.

1.3 The 4-velocity and 4-acceleration

If the position of an object is $\boldsymbol{x}$, then the velocity is

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}. \tag{1.3.1}$$

This expression suffers the same problems as the traditional form of Newton's second law, and we solve these problems in the same way. Eliminating dt in favor of $d\tau$, we get

$$\boldsymbol{u} \equiv \gamma \boldsymbol{v} = \frac{d\boldsymbol{x}}{d\tau} \tag{1.3.2}$$

where $\boldsymbol{u} = \gamma\boldsymbol{v}$ is the spacelike component of the *4-velocity* $\underline{u} = (\boldsymbol{u}, u_t) = (\gamma\boldsymbol{v}, \gamma)$. The timelike component is just

$$u_t = \frac{dt}{d\tau} = \gamma, \tag{1.3.3}$$

so

$$\underline{u} = \frac{d\underline{x}}{d\tau} \tag{1.3.4}$$

where the spacetime position vector is $\underline{x} = (\boldsymbol{x}, t)$ as usual. Note that $\underline{u}$ is truly a 4-vector, as its length is a scalar:

$$\underline{u} \cdot \underline{u} = \gamma^2 v^2 - \gamma^2 = -1. \tag{1.3.5}$$

It is also easy to show that the 4-velocity is related to the 4-momentum by

$$\underline{p} = m\underline{u}. \tag{1.3.6}$$

The 4-acceleration is a 4-vector defined as follows:

$$\underline{\alpha} = \frac{d\underline{u}}{d\tau}. \tag{1.3.7}$$

We know that it is a 4-vector, because the right side of the equation defining it is also a 4-vector. The 4-acceleration is normal to the 4-velocity. The Minkowski form of Newton's second law can be written

$$\underline{M} = m\underline{\alpha}. \tag{1.3.8}$$

In the rest frame, it may be shown that the 4-acceleration of some moving object is

$$\underline{\alpha} = \frac{d\underline{u}}{d\tau} = \left[\gamma^2\boldsymbol{a} + \gamma^4(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}, \gamma^4(\boldsymbol{v}\cdot\boldsymbol{a})\right] \tag{1.3.9}$$

where $\boldsymbol{v}$ and $\boldsymbol{a} = d\boldsymbol{v}/dt$ are the actual velocity and acceleration of the object in the rest frame. The form of the 4-acceleration depends very strongly as to whether the acceleration is parallel or normal to the pre-existing velocity. If the acceleration is parallel to the velocity, this reduces to

$$\underline{\alpha} = \left(\gamma^4\boldsymbol{a}, \gamma^4 va\right) \tag{1.3.10}$$

while for an acceleration normal to the velocity we have

$$\underline{\alpha} = \left(\gamma^2 \boldsymbol{a}, 0\right). \tag{1.3.11}$$

An inertially co-moving reference frame relative to a moving and possibly accelerating object is a frame moving at the velocity of the object at a specified instant in time, but not accelerating. In an inertially co-moving frame, the 4-velocity of the object is zero and the 4-acceleration is

$$\underline{\alpha} = (\boldsymbol{a}_0, 0) \tag{1.3.12}$$

where $\boldsymbol{a}_0$ is the ordinary acceleration of the object in this frame. Note that $\boldsymbol{a}_0$ is distinct from the above acceleration $\boldsymbol{a}$ in the rest frame. However, the Lorentz transformation can be used to rewrite the components of $\underline{\alpha}$ in the rest frame (or any other frame).

1.4 Assessment of different approaches

Newton's second law is used to compute the motion of objects assuming that the force is known. Both the traditional form of Newton's second law, as given by equation (1.2.1), and the Minkowski form, as given by equation (1.2.4), are useful under different circumstances. The traditional form is valid for tracking the relativistic motion of an object from a fixed rest frame in which the forces are known. For instance, the Lorentz force on a particle with charge q

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{1.4.1}$$

can be used to find the relativistic motion of such a particle relative to the fixed frame. However, transforming results to alternative frames is difficult, because the rules for transforming the ordinary force to a new frame are complicated.

The Minkowski force approach is best if one is interested in computing the motion of an object from the object's own perspective, since integrating equation (1.2.4) yields a result in terms of the proper time of the object τ rather then the time t of the fixed coordinate system. The covariant form of equation (1.2.4) also allows easy changes of reference frame, which is sometimes useful.

1.5 Electromagnetic force in covariant form

From our perspective it is easiest to write the electromagnetic force in covariant form using the scalar and vector potentials combined as a 4-potential than to do it in terms of the electric and magnetic fields. This is because the components of these fields actually form the components of a *4-tensor*, which carries us to a level of mathematics that is more complex than we desire at this point.

We recall that the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields are written in terms of the scalar ϕ and vector $\boldsymbol{A}$ potentials as follows:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{1.5.1}$$

1.5.1 Lorenz condition and the 4-gradient

Recall also that the scalar and vector potentials are not completely independent of each other, but are related by the Lorenz condition:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{\partial \phi}{\partial t} = 0 \tag{1.5.2}$$

where we again remind ourselves that we are setting $c = 1$. The scalar and vector potentials are really the timelike and spacelike components of a 4-potential

$$\underline{A} = (\boldsymbol{A}, \phi) \tag{1.5.3}$$

and the Lorenz condition can be written in invariant form as

$$\underline{\Box} \cdot \underline{A} = 0 \tag{1.5.4}$$

where

$$\underline{\Box} = \left(\boldsymbol{\nabla}, -\frac{\partial}{\partial t} \right) \tag{1.5.5}$$

is called the *4-gradient*. (The minus sign on the time derivative is needed to make $\underline{\Box}$ a 4-vector. Test this by taking the 4-gradient of a plane wave $\psi = \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]$; the result should be a 4-vector.)

1.5.2 Electromagnetic Minkowski force

Substituting equations (1.5.1) into equation (1.4.1) and using equation (1.2.2) gives us the spacelike part of the Minkowski force for electromagnetism:

$$\boldsymbol{M} = \gamma \boldsymbol{F} = \gamma q \left[-\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \right]. \tag{1.5.6}$$

From vector calculus we have

$$\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) - \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{A} \tag{1.5.7}$$

where it is important to realize that the derivatives in the gradient operator apply only to $\boldsymbol{A}$; the velocity $\boldsymbol{v}$ is a constant under spatial differentiation.

In general $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, t)$. We define a special kind of total time derivative of $\boldsymbol{A}$ which is the derivative following the position of the charged particle on which the electromagnetic force is acting. For the purposes of this derivative, we take $\boldsymbol{x} = \boldsymbol{x}_{particle}(t)$, so that $d\boldsymbol{x}/dt = \boldsymbol{v}$ where $\boldsymbol{v}$ is the particle velocity. Thus, the total time derivative of the vector potential, obtained using the chain rule, is

$$\frac{d\boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{A}. \tag{1.5.8}$$

Combining equations (1.5.6), (1.5.7), and (1.5.8) results in

$$\boldsymbol{M} = -\gamma q \frac{d\boldsymbol{A}}{dt} + q\boldsymbol{\nabla}(\gamma \boldsymbol{v} \cdot \boldsymbol{A} - \gamma\phi). \tag{1.5.9}$$

We recognize $\gamma \boldsymbol{v} \cdot \boldsymbol{A} - \gamma\phi$ as a dot product between the 4-velocity $\underline{u} = (\gamma \boldsymbol{v}, \gamma)$ and the 4-potential $A = (\boldsymbol{A}, \phi)$. Using also the fact that $dt = \gamma d\tau$, equation (1.5.9) simplifies to

$$\boldsymbol{M} = -q\frac{d\boldsymbol{A}}{d\tau} + q\boldsymbol{\nabla}(\underline{u} \cdot \underline{A}). \tag{1.5.10}$$

Extending this to the Minkowski 4-force is a simple matter of writing down a covariant expression for $\underline{M}$ that has a spacelike component given by equation (1.5.10).This is pretty clearly

$$\underline{M} = -q\frac{d\underline{A}}{d\tau} + q\Box(\underline{u} \cdot \underline{A}). \tag{1.5.11}$$

The final step is to note that the proper time derivative of 4-momentum in equation (1.2.4) can be combined with the first term on the right in equation (1.5.11), resulting in the very compact relativistic form of Newton's second law for electromagnetism

$$\frac{d\underline{\Pi}}{d\tau} \equiv \frac{d}{d\tau}\left(\underline{p} + q\underline{A}\right) = q\Box(\underline{u} \cdot \underline{A}). \tag{1.5.12}$$

The quantity

$$\underline{\Pi} = \underline{p} + q\underline{A} \tag{1.5.13}$$

is the total 4-momentum discussed in Physics 222. Equation (1.5.12) shows that the total 4-momentum of a particle is conserved if $\underline{u}$ is normal (in a spacetime sense) to $\underline{A}$.

1.6 Problems

1. Prove that the 4-velocity and the 4-acceleration of an object are normal or perpendicular in the relativistic sense.

2. Four-acceleration:

   (a) Prove equation (1.3.9).

   (b) Prove equation (1.3.10).

   (c) Prove equation (1.3.11).

3. An object of mass m is moving with velocity $\boldsymbol{v} = v\hat{\boldsymbol{\imath}}$ relative to a stationary frame at some instant and its ordinary acceleration in a co-moving inertial frame at that instant is $\boldsymbol{a}_0 = a_0\hat{\boldsymbol{\imath}}$ where a_0 is constant.

   (a) Find its 4-velocity at that instant in the stationary frame.

   (b) Find its 4-acceleration at that instant in the stationary frame. Hint: Make a Lorentz transformation of the 4-acceleration from the co-moving inertial frame to the stationary frame.

   (c) Show that the acceleration in the stationary frame is $a = a_0/\gamma^3$. Hint: Compare the above results for $\underline{a}$ resolved in the rest frame with equation (1.3.10).

(d) Integrate $dv/dt = a$ to get $v(t)$, assuming that $v(0) = 0$. (You will probably have to look up the integral.)

4. A spaceship of mass m is subject to an ordinary force $\boldsymbol{F} = F\hat{\boldsymbol{\imath}}$ where F is constant.

   (a) If the spaceship starts from rest at time $t = 0$, compute its velocity as a function of time, assuming that the force acts long enough for it to reach relativistic speeds.

   (b) Compare the time dependence of the velocity in this case with that obtained in the previous problem. For comparison purposes, define $a = F/m$.

   (c) Recall that $d\tau/dt = 1/\gamma$. Using the fact that γ can be written in terms of at, integrate this equation to find $\tau = \tau(t)$, thereby obtaining elapsed proper time on the spaceship relative to rest frame time. (You will probably have to look up the integral.) Obtain an approximate form of this equation for $at \gg 1$. Comment on relative aging rates of individuals on the spaceship relative to those at rest.

5. Imagine a particle with mass m and charge q in relativistic circular motion in the x-y plane under the influence of a magnetic field B in the z direction.

   (a) Relate the radial component of the Minkowski force to the radial component of the 4-acceleration using the covariant form of Newton's second law. Then eliminate the Minkowski force for the regular radial magnetic force, which has magnitude qvB as usual, where v is the tangential component of the particle velocity.

   (b) Eliminate the 4-acceleration in favor of the regular radial circulation, noting that this is a case in which the acceleration is normal to the velocity.

   (c) The radial acceleration still equals v^2/r in relativity, where r is the radius of the circle, since this is just geometry. Thus obtain the angular frequency of revolution $\omega = v/r$ in the relativistic case, and compare it to the non-relativistic case.