Chapter 1

Oscillations

It has been said that the progress of theoretical physics is marked by more and more sophisticated ways of solving the harmonic oscillator problem! There is no question that the harmonic oscillator plays a key role in both theoretical physics and in applications.

1.1 Prototypical oscillator – the simple pendulum

As we have shown, the simple pendulum with a mass on the bottom of a rod of negligible mass obeys the equation

$$\frac{d^2\phi}{dt^2} + \frac{g}{l}\sin\phi = 0 \tag{1.1}$$

where the length of the pendulum rod is l, ϕ is the angle of the rod from the vertical, and g is the gravitational field strength. We choose this rather than the mass-spring system because the pendulum exhibits a characteristic of many oscillating systems – the oscillation is only simple harmonic for small amplitudes. In the case of the pendulum, the oscillation period increases with amplitude, as we now show.

We demonstrate this using the energy equation. The total energy of the pendulum is

$$E = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\alpha \tag{1.2}$$

where α is the maximum value of ϕ (at which point $\dot{\phi} = 0$) for the given energy. Solving for $\dot{\phi}$ yields

$$\frac{d\phi}{dt} = \left(\frac{2g}{l}\right)^{1/2}(\cos\phi - \cos\alpha)^{1/2}. \tag{1.3}$$

We rewrite this as

$$\frac{d\phi}{(\cos\phi - \cos\alpha)^{1/2}} = \left(\frac{2g}{l}\right)^{1/2} dt \tag{1.4}$$

and integrate over a quarter period $T/4$, during which time ϕ goes from zero to α:

$$\int_0^\alpha \frac{d\phi}{(\cos\phi - \cos\alpha)^{1/2}} = \left(\frac{2g}{l}\right)^{1/2}\frac{T}{4} = \left(\frac{g}{8l}\right)^{1/2} T. \tag{1.5}$$

No closed form solution for the above integral exists. However, if we approximate $\cos\phi = 1 - \phi^2/2$ and $\cos\alpha = 1 - \alpha^2/2$, the integral can be performed:

$$\begin{aligned}\int_0^\alpha \frac{d\phi}{(\cos\phi - \cos\alpha)^{1/2}} \approx & \int_0^\alpha \frac{d\phi}{(\alpha^2/2 - \phi^2/2)^{1/2}} \\ =& \frac{2^{1/2}}{\alpha}\int_0^\alpha \frac{d\phi}{(1-\phi^2/\alpha^2)^{1/2}} \\ =& 2^{1/2}\int_0^1 \frac{dx}{(1-x^2)^{1/2}} \\ =& 2^{1/2} \times (\pi/2),\end{aligned} \tag{1.6}$$

from which we find the period to be

$$T = \left(\frac{8l}{g}\right)^{1/2} \times 2^{1/2} \times (\pi/2) = 2\pi\left(\frac{l}{g}\right)^{1/2}, \tag{1.7}$$

which is the classical period for small amplitude oscillations.

The next order of approximation is obtained by setting $\cos x \approx 1 - x^2/2 + x^4/24$, in which case we have

$$\begin{aligned}\int_0^\alpha \frac{d\phi}{(\cos\phi - \cos\alpha)^{1/2}} \approx & \int_0^\alpha \frac{d\phi}{(\alpha^2/2 - \phi^2/2 - \alpha^4/24 + \phi^4/24)^{1/2}} \\ =& 2^{1/2}\int_0^1 \frac{dx}{\left[1 - x^2 - (\alpha^2/12)(1-x^4)\right]^{1/2}} \\ =& 2^{1/2}\int_0^1 \frac{dx}{(1-x^2)^{1/2}\left[1 - (\alpha^2/12)(1+x^2)\right]^{1/2}}\end{aligned} \tag{1.8}$$

where we have used the fact that $1 - x^4 = (1-x^2)(1+x^2)$. The term in square brackets can be further approximated to order α^2 as follows:

$$\left[1 - (\alpha^2/12)(1+x^2)\right]^{-1/2} \approx 1 + (\alpha^2/24)(1+x^2). \tag{1.9}$$

The integral

$$\int_0^1 \frac{(1+x^2)dx}{(1-x^2)^{1/2}} = \frac{3\pi}{4},$$

so we finally get the period in this approximation

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2}\left(1 + \frac{\alpha^2}{16}\right), \tag{1.10}$$

which shows that the period increases with amplitude α is stated above.

1.2 Linearization of simple pendulum

Most oscillating systems in the real world are like the simple pendulum – they only oscillate harmonically (i.e., with frequency independent of amplitude) when the amplitude of the oscillation is small. It is useful to explore the small amplitude behavior of such oscillators. In this section we present the procedure for doing so.

The first step in this analysis is to find the steady state behavior of the system. In the case of the pendulum, this occurs when the angular acceleration $d^2\phi/dt^2 = 0$. From equation (1.1), we see that this results in $\sin\phi = 0$. There are two angles that satisfy this condition, $\phi_0 = 0, \pi$. Let us examine small amplitude motions about each of these equilibrium points. We do this by making a Taylor series expansion of $\sin\phi$ about the equilibrium point. Assuming that $\phi = \phi_0 + \phi'$, we find that

$$\sin(\phi_0 + \phi') \approx \sin(\phi_0) + \left.\frac{d\sin\phi}{d\phi}\right|_{\phi_0} \phi' = \sin\phi_0 + \cos\phi_0\phi' \tag{1.11}$$

where we have kept only the zeroth and first order terms in the expansion.

For $\phi_0 = 0$, $\sin(\phi_0 + \phi') \approx \phi'$ whereas for $\phi = \pi$, $\sin(\phi_0 + \phi') \approx -\phi'$. In the first case the governing equation (1.1) becomes

$$\frac{d^2\phi'}{dt^2} + \frac{g}{l}\phi' = 0, \qquad \phi_0 = 0, \tag{1.12}$$

whereas in the second case

$$\frac{d^2\phi'}{dt^2} - \frac{g}{l}\phi' = 0, \qquad \phi_0 = \pi. \tag{1.13}$$

These solutions have markedly different behavior. Rather than using sines and cosines, we solve these equations with an exponential function as a trial solution: $\phi' = \exp(\sigma t)$ where σ is a constant to be determined. For the $\phi_0 = 0$ case, substitution of this solution and cancellation of the exponential function results in the condition

$$\sigma = \pm i\left(\frac{g}{l}\right)^{1/2} \equiv \pm i\omega \qquad \phi_0 = 0 \tag{1.14}$$

so that we have in general a superposition of complex exponential solutions

$$\phi' = A\exp(i\omega t) + B\exp(-i\omega t) \tag{1.15}$$

where A and B are (possibly complex) constants. Using Euler's equations, this can be rewritten in terms of sines and cosines

$$\phi' = C\cos(\omega t) + D\sin(\omega t). \tag{1.16}$$

which means that the solution is oscillatory.

For $\phi_0 = \pi$, we have

$$\sigma = \pm\left(\frac{g}{l}\right)^{1/2} \qquad \phi_0 = \pi \tag{1.17}$$

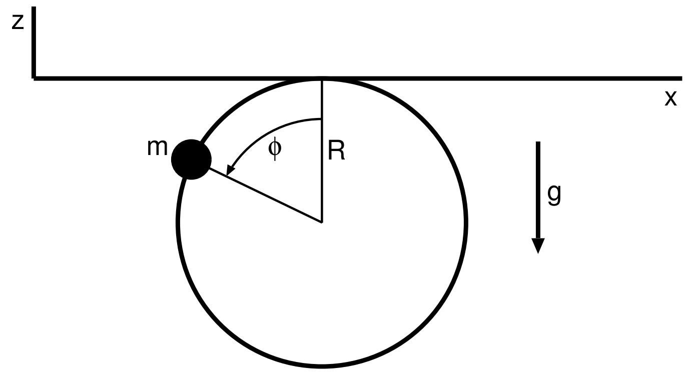

Figure 1.1: Huygens pendulum, consisting of a light ring with an attached mass which rolls on the underside of a flat surface.

and the solutions are real exponentials

$$\phi' = A\exp(\sigma t) + B\exp(-\sigma t). \tag{1.18}$$

One of these solutions decays away with time and the other grows indefinitely. Thus, the perturbation angle increases without bound, or at least until the linearization assumption fails. This system is *unstable*, whereas the previous case is *stable and oscillatory*.

The reasons for this behavior in the simple pendulum are obvious; if the mass starts out on top, minor deviations from the vertical orientation of the pendulum rod amplify with time under the influence of gravity, whereas if it starts on the bottom, these deviations will be opposed by gravity. This case forms a useful prototype for investigating the behavior of more complex systems.

1.3 The Huygens pendulum

At this point we can't pass up a description of the Huygens or cycloidal pendulum, which is presented in Sommerfeld. This pendulum, which was invented by Christian Huygens in 1673, causes the mass to move along a cycloidal curve rather than a circular arc. The actual mechanism used by Huygens to effect this behavior is described in Sommerfeld. However, Sommerfeld presents an alternative mechanism, illustrated in figure 1.1, that is easier to analyze, though harder to implement in hardware.

The pendulum consists of a lightweight ring of radius R with a mass m attached at one point on the ring, which is somehow (perhaps with a magnetic force) made to roll on the underside of a flat surface under the influence of gravity. The coordinates of the mass are readily determined to be

$$x = R(\phi - \sin\phi) \qquad z = -R(1 - \cos\phi) \tag{1.19}$$

where ϕ is the rotation angle of the ring, with the mass at the top for $\phi = 0$. The velocity components of the mass are obtained by differentiating equations (1.19) with respect to time,

$$\dot{x} = R(1 - \cos\phi)\dot{\phi} \qquad \dot{z} = -R\sin\phi\dot{\phi}, \tag{1.20}$$

from which the kinetic energy may be computed:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) = \frac{1}{2}mR^2\left[(1-\cos\phi)^2 + \sin^2\phi\right]\dot{\phi}^2 = mR^2\left(1-\cos\phi\right)\dot{\phi}^2. \tag{1.21}$$

The potential energy is just

$$V = mgz = -mgR(1-\cos\phi). \tag{1.22}$$

The angle ϕ turns out not to be the most useful generalized coordinate in this case. In order to discover this coordinate, we invoke the trig identity

$$1 - \cos\phi = 2\sin^2\frac{\phi}{2}, \tag{1.23}$$

resulting in

$$T = 2mR^2\sin^2\frac{\phi}{2}\dot{\phi}^2 = 8mR^2\left(\frac{d}{dt}\cos\frac{\phi}{2}\right)^2 \tag{1.24}$$

and

$$V = -2mgR\sin^2\frac{\phi}{2} = -2mgR\left(1-\cos^2\frac{\phi}{2}\right). \tag{1.25}$$

It is now obvious that $u \equiv \cos(\phi/2)$ is the appropriate coordinate, from which we infer that the Langrangian is

$$L = 8mR^2\dot{u}^2 + 2mgR(1-u^2). \tag{1.26}$$

Substitution into Lagrange's equation thus results in

$$\frac{d^2u}{dt^2} + \frac{g}{4R}u = 0. \tag{1.27}$$

This is just the harmonic oscillator equation with oscillation frequency $\omega = (g/4R)^{1/2}$ and generalized coordinate $u \equiv \cos(\phi/2)$.

1.4 Spherical pendulum

One of the problems in the chapter on Lagrange's equations derived the governing equations for the spherical pendulum, which is a mass m attached to a light rod of length l which is free to pivot in all directions at the end opposite the mass. The governing equations are

$$ml^2\left(\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\dot{\phi}^2\right) + mgl\sin\theta = 0 \tag{1.28}$$

and

$$\frac{dp_\phi}{dt} = \frac{d}{dt}\left(ml^2\sin^2\theta\dot{\phi}\right) = 0 \tag{1.29}$$

where ϕ is the azimuthal angle of the mass relative to its rest position in the horizontal plane, θ is the angle between the rod and the vertical, and

$$p_\phi = ml^2\sin^2\theta\dot{\phi} = constant \tag{1.30}$$

is the conserved generalized momentum associated with the generalized coordinate ϕ, in reality the vertical component of the angular momentum of the mass.

Eliminating $\dot{\phi}$ in favor of p_ϕ in equation (1.28) results in

$$\frac{d^2\theta}{dt^2} - \frac{\cos\theta}{\sin^3\theta}\left(\frac{p_\phi}{ml^2}\right)^2 + \frac{g}{l}\sin\theta = 0. \tag{1.31}$$

If the angular momentum is zero, the spherical pendulum just oscillates like a normal pendulum. However, if it is non-zero, then there is an equilibrium state in which the mass moves in a horizontally oriented circle of radius $l\sin\theta$, centered on the vertical line passing through the pivot point. Setting $d^2\theta/dt^2 = 0$ in equation (1.31) gives us a relation between the angular momentum and the angle θ. Solving this for θ is impractical, as the equilibrium equation is quartic in $\cos\theta$. However, we are free to specify the equilibrium value of $\theta = \theta_0$ and solve for the angular momentum required to produce this value:

$$p_\phi = \left(\frac{m^2 l^3 g \sin^4\theta_0}{\cos\theta_0}\right)^{1/2}. \tag{1.32}$$

From equation (1.30) we find that azimuthal angular velocity of this rotation is

$$\omega \equiv \dot{\phi} = \left(\frac{g}{l\cos\theta_0}\right)^{1/2}, \tag{1.33}$$

which we have renamed ω. Notice that for small amplitude, $\cos\theta_0 \approx 1$ and ω becomes the classical oscillation frequency of a simple pendulum. The circular motion may be considered to be a superposition of linear oscillations in the two orthogonal horizontal directions that are out of phase by 90°, so it is no surprise that the frequency of circular motion is the same as that for linear oscillations in the small amplitude limit. It is interesting that the angular frequency for circular oscillations increases with amplitude, whereas the frequency for linear oscillations decreases with amplitude.

We now investigate small oscillations in θ about this state of circular motion by setting $\theta = \theta_0 + \theta'$ and linearizing equation (1.31) in θ' while retaining the constancy of p_ϕ. To do so we must make first order Taylor series expansions of $-\cos\theta/\sin^3\theta$ and $\sin\theta$ about θ_0. The results are

$$-\frac{\cos\theta}{\sin^3\theta} \approx -\frac{\cos\theta_0}{\sin^3\theta_0} + \left(\frac{1+2\cos^2\theta_0}{\sin^4\theta_0}\right)\theta' \tag{1.34}$$

and

$$\sin\theta \approx \sin\theta_0 + \cos\theta_0\theta' \tag{1.35}$$

which yields

$$\left[\frac{d^2\theta_0}{dt^2} - \frac{\cos\theta_0}{\sin^3\theta_0}\left(\frac{p_\phi}{ml^2}\right)^2 + \frac{g}{l}\sin\theta_0\right] +$$
$$\left[\frac{d^2}{dt^2} + \frac{1+2\cos^2\theta_0}{\sin^4\theta_0}\left(\frac{p_\phi}{ml^2}\right)^2 + \frac{g}{l}\cos\theta_0\right]\theta' \approx 0 \tag{1.36}$$

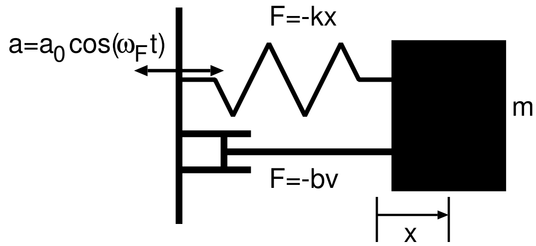

Figure 1.2: Harmonic oscillator that is forced by a periodic motion on the left end of the spring and damped by a dashpot (shock absorber) in parallel with the spring.

upon substitution into equation (1.31). Since θ_0 is a constant, time second time derivative of θ_0 vanishes and the quantity in the first set of brackets in equation (1.36) is zero by virtue of the value of p_ϕ in equation (1.32).

Eliminating p_ϕ in the second set of brackets using equation (1.32) and simplfying results in

$$\frac{d^2\theta'}{dt^2} + \left[\frac{1+3\cos^2\theta_0}{\cos\theta_0}\left(\frac{g}{l}\right)\right]\theta' = 0 \tag{1.37}$$

which we identify as a harmonic oscillator equation for oscillations in the θ direction with frequency

$$\Omega = \left[\frac{1+3\cos^2\theta_0}{\cos\theta_0}\left(\frac{g}{l}\right)\right]^{1/2} = (1+3\cos^2\theta_0)^{1/2}\omega. \tag{1.38}$$

Note that for small circles in which $\cos\theta_0 \approx 1$, we have $\Omega \approx 2\omega$. Thus, there are two maxima and two minima in θ for every revolution of the mass about the vertical axis. Another way of looking at this case is as a superposition of orthogonal linear oscillations with one of these having greater amplitude than the other, resulting in an elliptical rather than a circular motion in the horizontal plane.

As θ_0 increases, the cosine function becomes smaller, the ratio of Ω to ω becomes less than 2, and the major axis of the ellipse precesses in the direction of rotation. Sommerfeld illustrates this effect in his book.

1.5 Forced dissipative oscillators

So far we have considered only oscillators that are free and undamped. Figure 1.2 shows a forced mass-spring system with a shock absorber that resists the motion of the mass with a force proportional to the mass's velocity, $F = -b\dot{x}$, where b is a constant. The forcing is provided by periodically varying the position of the left end of the spring (but not the shock absorber) by an amount equal to $a = a_0\cos(\omega_F t)$, where a_0 is the amplitude of the forcing and ω_F is its frequency. The damping force is non-conservative, so it must appear on the right side of Lagrange's equation. The Lagrangian is relatively simple and is just

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x-a)^2 \tag{1.39}$$

and the equation of motion is

$$m\frac{d^2x}{dt^2} + bx + kx = ka = ka_0 \cos(\omega_F t). \tag{1.40}$$

In order to solve this equation, let us first consider the alternative equation

$$m\frac{d^2z}{dt^2} + bz + kz = ka = ka_0 \exp(i\omega_F t). \tag{1.41}$$

If x is the real part of the complex variable z, then, since ω_F, m, b, k, and a_0 are real, then it is clear that equation (1.40) is just the real part of equation (1.41). One way to solve equation (1.40) is therefore to solve equation (1.41) and take the real part of the solution. The advantage to this procedure is that the solution to equation (1.40) is a complicated sum of sines and cosines, whereas the solution to equation (1.41) is a single exponential function.

Equation (1.41) is linear and inhomogeneous in z. The general solution is the superposition of any solution to the full, inhomogeneous equation plus the general solution to the homogeneous equation with the right side set to zero. Considering the homogeneous case first, let us assume that $z \propto \exp(i\omega t)$ where ω is to be determined. Substituting this into equation (1.41) with the right side zero results in

$$\omega^2 - \frac{ib}{m}\omega - \frac{k}{m} = 0. \tag{1.42}$$

Solving this quadratic equation for ω gives us

$$\omega = \frac{1}{2}\left[\frac{ib}{m} \pm \left(-\frac{b^2}{m^2} + \frac{4k}{m}\right)^{1/2}\right] = \pm\left(\frac{k}{m} - \frac{b^2}{4m^2}\right)^{1/2} + \frac{ib}{m}, \tag{1.43}$$

from which we find

$$z = z_0 \exp\left\{\left[\pm i\left(\frac{k}{m} - \frac{b^2}{4m^2}\right)^{1/2} - \frac{b}{m}\right]t - i\phi\right\} \tag{1.44}$$

where z_0 is a real constant and ϕ is a phase factor needed to make this solution general.

Taking the real part gives us the physical homogeneous solution

$$x_H = z_0 \exp\left(-\frac{bt}{m}\right) \cos\left[\left(\frac{k}{m} - \frac{b^2}{4m^2}\right)^{1/2} t - \phi\right] \tag{1.45}$$

which is in the form of a decaying exponential function modulated by an oscillating cosine function of time.

The oscillation frequency is less than the classical undamped mass-spring system $(k/m)^{1/2}$ and asymptotes to it as $b \to 0$. On the other hand, if the quantity inside the square root in equation (1.45) is negative, the solution is a superposition of two exponential functions that decay at different rates

$$x_H = z_1 \exp\left[-\frac{bt}{m} + \left(\frac{b^2}{4m^2} - \frac{k}{m}\right)^{1/2} t\right] + z_2 \exp\left[-\frac{bt}{m} - \left(\frac{b^2}{4m^2} - \frac{k}{m}\right)^{1/2} t\right]. \tag{1.46}$$

where z_1 and z_2 are arbitrary constants.

A special case arises when the quantity inside the square root is zero, so that

$$\frac{k}{m} = \frac{b^2}{4m^2}. \tag{1.47}$$

in this case the two terms in equation (1.46) are not independent and a second solution needs to be sought. It is readily shown that the full solution in this case takes the form

$$x = (z_0 + v_0 t) \exp\left(-\frac{bt}{m}\right) \tag{1.48}$$

where z_0 and v_0 are arbitrary constants.

We turn now to obtaining a particular solution to the full inhomogeneous equation (1.41). The simplest assumption to make is that $z = z_0 \exp(i\omega_F t)$, where z_0 is now possibly complex. The resulting algebraic relation after cancellation of the exponential function is

$$\left(-m\omega_F^2 + ib\omega_F + k\right) z_0 = ka_0. \tag{1.49}$$

Unlike the homogeneous case, the frequency is the specified forcing frequency and the unknown is z_0. Solving for this quantity results in

$$z = \frac{a_0 \omega_0^2 \exp(i\omega_F t)}{\omega_0^2 - \omega_F^2 + i\beta\omega_0\omega_F} \tag{1.50}$$

where $\omega_0^2 = k/m$ is the square of the undamped free oscillator and $\beta = b/(m\omega_0)$. The complex denominator may be written in polar form as

$$\omega_0^2 - \omega_F^2 + i\beta\omega_0\omega_F = \left[\left(\omega_0^2 - \omega_F^2\right)^2 + \beta^2\omega_0^2\omega_F^2\right]^{1/2} \exp(i\phi_F) \tag{1.51}$$

where the phase angle is

$$\phi_F = \tan^{-1}\left(\frac{\beta\omega_0\omega_F}{\omega_0^2 - \omega_F^2}\right). \tag{1.52}$$

The solution (1.50) can therefore be written

$$z = \frac{a_0\omega_0^2 \exp\left[i\left(\omega_F t - \phi_F\right)\right]}{\left[\left(\omega_0^2 - \omega_F^2\right)^2 + \beta^2\omega_0^2\omega_F^2\right]^{1/2}}. \tag{1.53}$$

Taking the real part gives us the physical solution to the inhomogeneous problem

$$x_I = \frac{a_0\omega_0^2 \cos\left(\omega_F t - \phi_F\right)}{\left[\left(\omega_0^2 - \omega_F^2\right)^2 + \beta^2\omega_0^2\omega_F^2\right]^{1/2}}. \tag{1.54}$$

This expression may be simplified by introducing the dimensionless forcing frequency $\Omega_F = \omega_F/\omega_0$, the dimensionless time $\tau = \omega_0 t$, and the dimensionless displacement χ_I:

$$\chi_I = \frac{\cos(\Omega_F\tau - \phi_F)}{\left[(1-\Omega_F^2)^2 + \beta^2\right]^{1/2}} \qquad \phi_F = \tan^{-1}\left(\frac{\beta}{1-\Omega_F^2}\right) \tag{1.55}$$

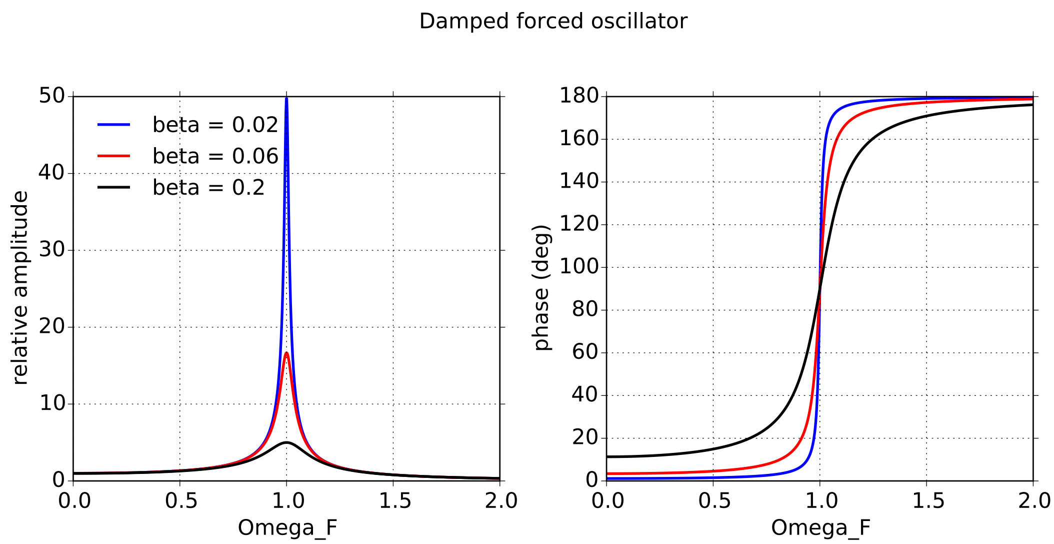

Figure 1.3: Amplitude of oscillation relative to forcing (left) and the phase of the response as a function of the dimensionless forcing frequency.

With similar scaling, the homogeneous solution (1.45) may be written

$$\chi_H = \chi_0 \exp(-\beta\tau) \cos\left[\left(1-\beta^2/2\right)^{1/2} \tau - \phi\right] \tag{1.56}$$

where χ_0 is an arbitrary dimensionless constant. Such non-dimensionalization de-clutters equations and makes them easier to understand. Figure 1.3 shows the amplitude of the response relative to the forcing of the damped oscillator (left panel) and the phase (right panel), as expressed in equation (1.55).

The total solution is the sum of the homogeneous and inhomogeneous parts. In practice it is often sufficient to consider only the inhomogeneous part of the solution, as the homogeneous part decays with time and eventually dies out. However, both parts must be considered, and the constants (z_0, ϕ) or (z_1, z_2) adjusted appropriately, in the case of an initial value problem.

1.6 Coupled oscillators

In the previous chapter's problems, we considered the response of a system consisting of two masses and two springs in series to sinusoidal forcing. Changing the notation slightly, the governing equations are

$$\begin{aligned} \frac{1}{\omega_0^2}\frac{d^2x}{dt^2} + 2x - y =& a_0 \exp(i\omega_F t) \\ \frac{1}{\omega_0^2}\frac{d^2y}{dt^2} + y - x =& 0 \end{aligned} \tag{1.57}$$

where x and y are the deviations of the masses from their equilibrium conditions with the forcing turned off, i.e., with $a_0 = 0$. The quantity $\omega_0 = (k/m)^{1/2}$ is the frequency of oscillation of a single mass-spring system in isolation. We have put the equations in complex form such that taking the real parts brings us back to the original equations. Since the equations are linear, this is a legitimate process, and the solutions are easier to obtain.

We now write equations (1.57) in matrix form with the additional change of variables $\omega_0 t = \tau$, $(x/a_0, y/a_0) = (X, Y)$:

$$\frac{d^2}{d\tau^2}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp(i\Omega_F\tau) \tag{1.58}$$

where $\Omega_F = \omega_F/\omega_0$. The solution to this equation is composed of two parts, a particular inhomogeneous solution plus the general solution to the homogeneous problem, i.e., with the right side set to zero. Let us consider the latter solution first. Assuming that (X, Y) are proportional to $\exp(i\Omega\tau)$ where Ω is a dimensionless frequency scaled by ω_0, this becomes the eigenvalue problem

$$\begin{pmatrix} 2-\Omega^2 & -1 \\ -1 & 1-\Omega^2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0. \tag{1.59}$$

The equation has non-trivial solutions only when the determinant of the matrix equals zero:

$$\Omega^4 - 3\Omega^2 + 1 = 0. \tag{1.60}$$

The roots of this equation are the eigenvalues

$$\Omega_1^2 = \frac{3+5^{1/2}}{2} = 2.6180 \qquad \Omega_2^2 = \frac{3-5^{1/2}}{2} = 0.3820 \tag{1.61}$$

with corresponding eigenvector components determined by

$$(2-\Omega_{1,2}^2)X_{1,2} - Y_{1,2} = 0. \tag{1.62}$$

The normalized eigenvectors are

$$(X_{1,2}, Y_{1,2}) = \frac{(1, 2-\Omega_{1,2}^2)}{\left[1+(2-\Omega_{1,2}^2)^2\right]^{1/2}} \tag{1.63}$$

and the orthogonal transformation that diagonalizes the matrix

$$D = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \tag{1.64}$$

in equation (1.58) is

$$U = \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix} = \begin{pmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8587 \end{pmatrix}. \tag{1.65}$$

Thus

$$UDU^T = \begin{pmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{pmatrix} \tag{1.66}$$

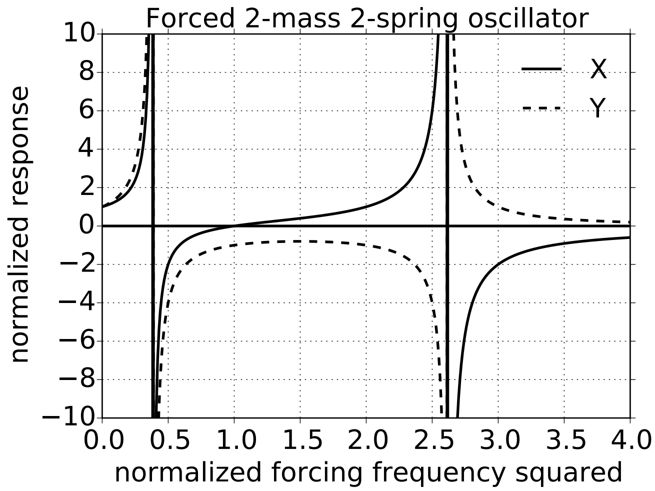

Figure 1.4: Response of mass-spring system with 2 degrees of freedom to oscillatory forcing.

where U^T is the transpose of U.

We now use these results to solve the inhomogeneous problem by multiplying equation (1.58) by U and inserting U^TU just to the right of the matrix D. This is legal because U^TU is just the identity matrix. Defining

$$U\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}, \tag{1.67}$$

equation (1.58) becomes

$$\frac{d^2}{d\tau^2}\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\exp(i\Omega_F\tau). \tag{1.68}$$

The importance of this method is that by working in terms of the amplitudes A and B of the eigenmodes of the unforced system instead of X and Y, the coupling between the two degrees of freedom of the system disappears. Dropping the matrix notation, equation (1.68) just becomes

$$\begin{aligned} \frac{d^2A}{d\tau^2} + \Omega_1^2 A =& X_1\exp(i\Omega_F\tau) \\ \frac{d^2B}{d\tau^2} + \Omega_2^2 B =& Y_1\exp(i\Omega_F\tau) \end{aligned} \tag{1.69}$$

i.e., we have two separate forced oscillator equations, each of which can be solved by the methods of the previous section. Once the solution is obtained, the original displacements of each mass can be obtained by inverting equation (1.67):

$$\begin{pmatrix} X \\ Y \end{pmatrix} = U^T\begin{pmatrix} A \\ B \end{pmatrix}. \tag{1.70}$$

Figure 1.4 shows X and Y as a function of the square of the dimensionless forcing frequency, Ω_F^2.

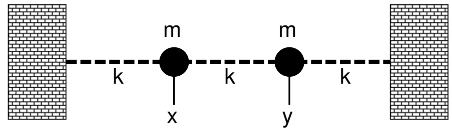

Figure 1.5: Coupled oscillator consisting of three springs connected to two masses.

1.7 Problems

1. Linearize the governing equation for the disk rolling around the inside of the cylinder introduced in problem 3 of the chapter on Lagrange's equations and find the small amplitude behavior of the system about the two equilibrium points. (Assume that the disk somehow sticks to the inside of the cylinder even when it is at the top.) The governing equation is

$$\frac{3}{2}M(b-a)^2\frac{d^2\theta}{dt^2} + Mg(b-a)\sin\theta = 0.$$

2. Three spring problem:

   (a) Find the Lagrangian for two masses m connected to two walls by three springs with spring constants k. The variables x and y are the deviations of the positions of the masses from their equilibrium points. The spacing of the walls is such that all springs are their unstretched or compressed lengths at equilibrium. The masses are constrained to move back and forth in only the horizontal direction.

   (b) Find the governing equations by the usual means and non-dimensionalize them using $\omega_0 t = \tau$ where ω_0 is the oscillation frequency of a single mass-spring system with the values given.

   (c) Write your equations in matrix form.

   (d) Seek oscillating solutions to the system by assuming that $x, y \propto \exp(i\Omega\tau)$ and find the eigenvalues (Ω^2) and eigenvectors of the system.

   (e) Using the normalized eigenvectors to create an orthogonal transformation matrix, transform the matrix equations to a form in which the two oscillation modes are decoupled.

   (f) Describe physically the oscillation modes that occur in this system.

3. Generalized Kepler problem:

   (a) Derive the Lagrangian expressed in cylindrical coordinates (r, ϕ) and from this, the governing equations for a mass m orbiting around mass $M \gg m$ under the influence of a potential of the form $V(r) = -Ar^{-n}$, where A and n are constants with $n > 0$. Show that the angular momentum p_ϕ is conserved and eliminate $\dot{\phi}$ in the r equation in favor of p_ϕ.

(b) Find a relationship between p_ϕ, A, and the radius of the circular orbit r_0 for the case in which $d^2r/dt^2 = 0$.

(c) Set $r = r_0 + r'$ and linearize the r equation about r_0, keeping p_ϕ fixed. Once linearized, use the results of (b) to eliminate the A term in favor of p_ϕ. Finally, eliminate p_ϕ in favor of r_0 and the angular frequency of revolution $\dot{\phi} \equiv \omega$. From the resulting equation, find the angular frequency of radial oscillations of the mass m as it revolves about mass M.

(d) Does the above result make qualitative sense in light of Kepler's second law for the case of gravitation?

(e) Are there values of n for which the circular orbit is unstable to small radial perturbations?

4. Parametric oscillator:

(a) Write down the Lagrangian for a simple pendulum in which the length l of the string supporting the mass m varies with time where this variation is externally imposed. Show that the governing equation in terms of deflection angle ϕ for small ϕ is

$$\frac{d^2\phi}{dt^2} + \omega^2\phi = -\frac{2\dot{l}\dot{\phi}}{l}$$

where $\omega^2 = g/l$.

(b) Derive the energy equation for this system by multiplying the above equation by $\dot{\phi}$ and rearranging so that

$$\frac{d\varepsilon}{dt} = -\frac{2\dot{l}\dot{\phi}^2}{l}$$

where the "energy" (actually the energy divided by ml^2) is

$$\varepsilon = \frac{\dot{\phi}^2 + \omega^2\phi^2}{2}.$$

(c) If $|\dot{l}/l| \ll 1$, the motion of the pendulum is nearly harmonic with a slowly changing amplitude, so that we can write $\phi = \phi_0 \sin(\omega t)$ to a good approximation. Show that $\phi^2 = \phi_0^2[1 - \cos(2\omega t)]/2$ and $\dot{\phi}^2 = \omega^2\phi_0^2[1 + \cos(2\omega t)]/2$. From this find ε in terms of ϕ_0 and ω and show that the energy equation can be written

$$\frac{1}{\varepsilon}\frac{d\varepsilon}{dt} = \frac{d\ln\varepsilon}{dt} = -\frac{2\dot{l}}{l}[1 + \cos(2\omega t)]$$

with integral

$$\ln(\varepsilon/\varepsilon_0) = -\int_0^t \frac{2\dot{l}}{l}[1 + \cos(2\omega t')]dt'.$$

(d) Given the above results, during which part of the oscillation should $\dot{l}$ be positive and which negative for ε to increase with time? Explain. To test your ideas, try integrating this equation over one full period of the pendulum oscillation with (where η is a constant)

i. $\dot{l}/l = -\eta \sin(\omega t)$;
ii. $\dot{l}/l = -\eta \cos(\omega t)$;
iii. $\dot{l}/l = -\eta \sin(2\omega t)$;
iv. $\dot{l}/l = -\eta \cos(2\omega t)$.