Chapter 1

Lagrange's equations

Starting with d'Alembert's principle, we now arrive at one of the most elegant and useful formulations of classical mechanics, generally referred to as Lagrange's equations. This treatment is taken from Goldstein's graduate mechanics text, as his treatment seems somewhat more clear to me than Sommerfeld's.

1.1 Lagrange's equations from d'Alembert's principle

We begin with d'Alembert's principle written in its most fundamental and general form,

$$\sum_i (F_i + F_i^*)\delta x_i = 0 \tag{1.1}$$

where the subscript i ranges over all three components of all particles involved in the system of interest. The first step is to rewrite the particle positions, represented by the x_i in groups of three for each particle, in terms of independent *generalized coordinates* q_j. If there are constraints in the system, then there are fewer q variables than x variables. For example, a wheel rotating on a fixed axle has only one q, the angle of rotation, whereas there are three times as many x variables as there are atoms in the wheel.

For holonomic constraints we can write

$$x_i = x_i(q_j, t) \tag{1.2}$$

where we allow for the possibility that the relationship between the q and x variables to depend on time.

We can rewrite d'Alembert's principle by noting that

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j \tag{1.3}$$

where the time dependence is not exercised since virtual changes are assumed to take place at a fixed time. Thus,

$$\sum_i F_i \delta x_i = \sum_{i,j} F_i \frac{\partial x_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \tag{1.4}$$

where the

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} \tag{1.5}$$

are called the *generalized forces*. Notice that just as the q_i need not have the units of length, the Q_i need not have the units of force. However, the product must have the units of energy. For instance, if q is a dimensionless angle, then the corresponding Q would be a torque, which has energy units.

Turning to the inertial forces in d'Alembert's principle, we note that

$$\sum_i F_i^* \delta x_i = -\sum_i m_i \frac{d^2 x_i}{dt^2} \delta x_i = -\sum_{i,j} m_i \frac{d^2 x_i}{dt^2} \frac{\partial x_i}{\partial q_j} \delta q_j \tag{1.6}$$

where we have used equation (1.3) in the last step. Using the product rule backwards, we see that

$$\sum_i m_i \frac{d^2 x_i}{dt^2} \frac{\partial x_i}{\partial q_j} = \sum_i m_i \left[\frac{d}{dt} \left(\dot{x}_i \frac{\partial x_i}{\partial q_j} \right) - \dot{x}_i \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_j} \right) \right] \tag{1.7}$$

To make further progress, we take the total time derivative of equation (1.2), resulting in

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t} \tag{1.8}$$

where $\dot{x}_i = dx_i/dt$ and $\dot{q}_j = dq_j/dt$. The $\dot{x}_i$ are the actual particle velocity components. We call the $\dot{q}_j$ the *generalized velocities*. Taking the partial derivative of equation (1.8) with respect to a particular q_j, we immediately conclude that

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}, \tag{1.9}$$

where the derivative of the second term in equation (1.8) is zero because the velocities are not functions of the positions at an instant in time. (Ultimately, the positions can be derived from the velocities by integration, but this relationship depends on knowing the complete time history of the velocities, not just the values at a particular time.)

Finally, we note that

$$\frac{d}{dt} \left(\frac{\partial x_i}{\partial q_j} \right) = \frac{\partial \dot{x}_i}{\partial q_j}. \tag{1.10}$$

To show this, change the dummy summation index in equation (1.8) from j to k to avoid confusion, and take the partial derivative of this equation with respect to q_j:

$$\frac{\partial \dot{x}_i}{\partial q_j} = \sum_k \frac{\partial^2 x_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 x_i}{\partial q_j \partial t}. \tag{1.11}$$

This is possible again because $\dot{q}_k$ is not an explicit function of the q_j. Then compare this with

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 x_i}{\partial t \partial q_j}. \tag{1.12}$$

Aside from the order of partial derivatives, the right sides of equations (1.11) and (1.12) are identical, which proves equation (1.10).

Substituting equations (1.9) and (1.10) in equation (1.7) results in

$$\begin{aligned}\sum_i m_i \frac{d^2 x_i}{dt^2}\frac{\partial x_i}{\partial q_j} =& \sum_i m_i \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) - \dot{x}_i\left(\frac{\partial \dot{x}_i}{\partial q_j}\right)\right] \\ =& \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\sum_i \frac{m_i \dot{x}_i^2}{2} - \frac{\partial}{\partial q_j}\sum_i \frac{m_i \dot{x}_i^2}{2} \\ =& \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \end{aligned} \tag{1.13}$$

where we recognize the sums as the total kinetic energy T of the system.

Combining equations (1.4), (1.6), and (1.13) yields

$$\sum_j \left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right]\delta q_j = 0. \tag{1.14}$$

Since the q_i are independent of each other, the coefficients of the δq_i are individually zero, resulting in Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{1.15}$$

Often forces are conservative and possible to represent as the gradient of a potential energy $F_i = -\partial V/\partial x_i$. Starting with the definition of generalized forces in equation (1.5), we find that

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \tag{1.16}$$

If in addition, V is not an explicit function of time or of the generalized velocities, equation (1.15) may be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.17}$$

where $L = T - V$ is called the *Lagrangian.* The lack of dependence on time and the generalized velocities allows the V to be incorporated in the first as well as the second terms of this equation. If some of the forces are conservative and others are not, then the more general form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \tag{1.18}$$

may be used.

The quantities

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{1.19}$$

are called the *generalized momenta*. Note that when the Lagrangian is not a function of a particular generalized coordinate and the associated non-conservative force Q_j is zero, then the associated generalized momentum is conserved, since equation (1.18) reduces to

$$\frac{dp_j}{dt} = 0. \tag{1.20}$$

To summarize, these equations are valid for systems obeying the following conditions:

1. The constraints on the system are holonomic, so that the q_j are independent for both finite and infinitesimal displacements. The constraints may be time dependent.

2. The potential energy V is a function only of the q_j. If there are forces for which no such potential exists, then they can be included on the right side of equation (1.18) in the Q_j.

1.2 Examples of use

We now look at several examples to see how Lagrange's equations are used.

1.2.1 Simple pendulum

We start with the simple pendulum, a problem that is easily solvable by elementary methods. Imagine the pendulum arm to have negligible mass and length D. A mass M is attached to the one end of the arm and the other is attached to a support that allows the pendulum to pivot freely in the x-z plane. The angle between the pendulum arm and the vertical is ϕ.

Taking zero elevation as the pivot point, the potential energy of the pendulum mass is

$$V = Mgz = -MgD\cos\phi. \tag{1.21}$$

Motion in the x-z plane is constrained to be in the form of a circular arc of radius D, which means that the kinetic energy of the pendulum is

$$T = \frac{1}{2}MD^2\dot{\phi}^2, \tag{1.22}$$

which means that the Lagrangian function is

$$L(\phi, \dot{\phi}) = T - V = \frac{1}{2}MD^2\dot{\phi}^2 + MgD\cos\phi. \tag{1.23}$$

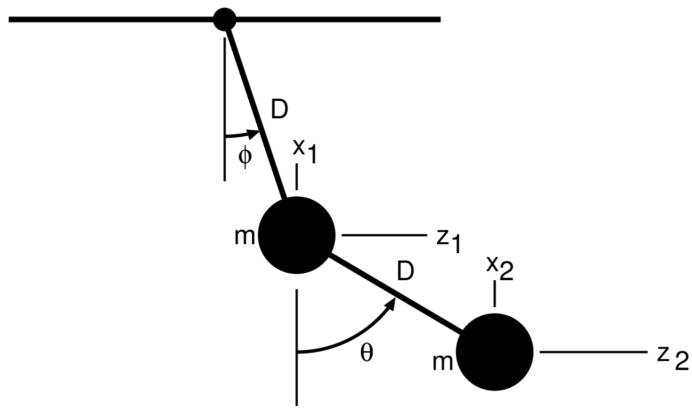

Figure 1.1: Sketch for the double pendulum.

The generalized coordinate (only one, as there is only one effective degree of freedom) is ϕ. Recall that ϕ and $\dot{\phi}$ are considered to be independent variables in Lagrangian dynamics, so

$$\frac{\partial L}{\partial \dot{\phi}} = MD^2\dot{\phi} \tag{1.24}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = MD^2\frac{d^2\phi}{dt^2}. \tag{1.25}$$

The second term in Lagrange's equation is also easily calculated:

$$\frac{\partial L}{\partial \phi} = -MgD\sin\phi. \tag{1.26}$$

Combining equations (1.25) and (1.26), we arrive at the governing equation for the pendulum:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = MD^2\frac{d^2\phi}{dt^2} + MgD\sin\phi = 0 \tag{1.27}$$

which simplifies to the usual form

$$\frac{d^2\phi}{dt^2} + \frac{g}{D}\sin\phi = 0. \tag{1.28}$$

For small ϕ we have $\sin\phi \approx \phi$ and we have a harmonic oscillator equation for the pendulum.

1.2.2 Double pendulum

A particular form of the double pendulum is illustrated in figure 1.1. The masses are free to swing in the x-z plane, with the second pendulum swinging from the bob on the first pendulum. (Other forms include a version with massive rods instead of weights attached to rods of negligible mass.)

The Cartesian coordinates of the two masses are related to the angles ϕ and θ as follows

$$(x_1, z_1) = (D\sin\phi, -D\sin\phi) \tag{1.29}$$

and

$$(x_2, z_2) = [D(\sin\phi + \sin\theta), -D(\cos\phi + \cos\theta) \tag{1.30}$$

where the origin of the coordinate system is located where the pendulum attaches to the ceiling. The kinetic energies of the two pendulums are

$$\begin{aligned} T_1 =& \frac{1}{2}m(\dot{x}_1^2 + \dot{z}_1^2) \\ =& \frac{1}{2}mD^2\dot{\phi}^2 \end{aligned} \tag{1.31}$$

$$\begin{aligned} T_2 =& \frac{1}{2}m(\dot{x}_2^2 + \dot{z}_2^2) \\ =& \frac{1}{2}mD^2\left[\dot{\phi}^2 + \dot{\theta}^2 + 2\cos(\phi - \theta)\dot{\phi}\dot{\theta}\right] \end{aligned} \tag{1.32}$$

and the potential energy of the two pendulum bobs together is

$$V = mg(z_1 + z_2) = -mgD\left(2\cos\phi + \cos\theta\right). \tag{1.33}$$

The angles ϕ and θ and their time derivatives are the generalized coordinates and velocities and the Lagrangian is

$$L = \frac{1}{2}mD^2\left[2\dot{\phi}^2 + \dot{\theta}^2 + 2\cos(\phi - \theta)\dot{\phi}\dot{\theta}\right] + mgD\left(2\cos\phi + \cos\theta\right). \tag{1.34}$$

From

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \tag{1.35}$$

we get the governing equations

$$2\frac{d^2\phi}{dt^2} + \cos(\phi - \theta)\frac{d^2\theta}{dt^2} + \sin(\phi - \theta)\dot{\theta}^2 + \frac{2g}{D}\sin\phi = 0 \tag{1.36}$$

and

$$\frac{d^2\theta}{dt^2} + \cos(\phi - \theta)\frac{d^2\phi}{dt^2} - \sin(\phi - \theta)\dot{\phi}^2 + \frac{g}{D}\sin\theta = 0. \tag{1.37}$$

Thus, two degrees of freedom result in two coupled governing equations. These happen to be complicated nonlinear equations and the physical system demonstrates chaotic behavior for large amplitude oscillations. Behavior at small amplitudes is more benign and will be investigated in the next chapter.

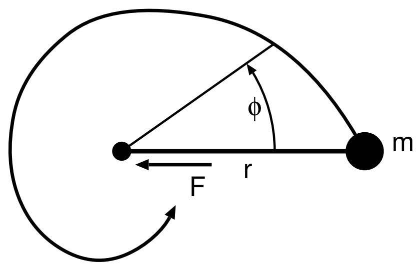

Figure 1.2: Schematic of the motion of a puck on an air table constrained by a string to which a force F is applied.

1.2.3 Air table problem

We now analyze a problem with a non-conservative, externally applied force. Imagine a puck of mass m sliding frictionlessly on a horizontal air table with a string attached that passes through a hole in the air table. A force F that may be varied at will is applied to the string, constraining the puck to move in some orbit around the hole.

There is no potential energy in this problem, so the Lagrangian is simply the kinetic energy. Referring to figure 1.2,

$$L = T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2. \tag{1.38}$$

The generalized force in the $+r$ direction is just $Q_r = -F$, since a positive F is toward the origin. There are two degrees of freedom for this problem, r and ϕ. The r component of Lagrange's equation is therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\frac{d^2 r}{dt^2} - mr\dot{\phi}^2 = -F \tag{1.39}$$

and the ϕ component is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0. \tag{1.40}$$

Notice that $\partial L/\partial\phi = 0$, with the consequence that the generalized momentum $p_\phi = mr^2\dot{\phi}$ is constant in time, i.e., it is conserved. We identify this quantity as the angular momentum about an axis of rotation normal to the air table and located at the hole in the table. Solving for $\dot{\phi}$ and substituting in equation (1.39) results in

$$m\frac{d^2 r}{dt^2} - \frac{p_\phi^2}{mr^3} = -F. \tag{1.41}$$

In equilibrium with constant r, this reduces to an equation for F,

$$F = mr\dot{\phi}^2, \tag{1.42}$$

which we recognize as the centripetal force for circular motion.

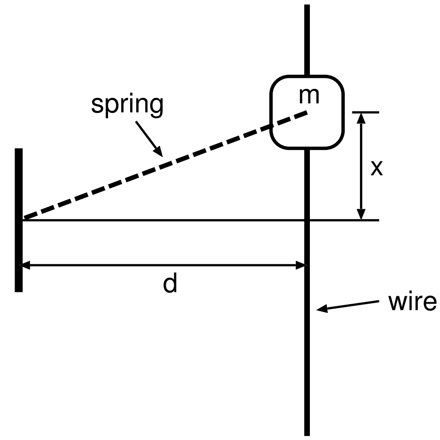

Figure 1.3: Sketch of a mass moving along a wire with a spring force.

This is an example of a general phenomenon with Lagrangian dynamics: if the Lagrangian doesn't depend on a particular generalized coordinate, in this case ϕ, then there exists a conserved quantity which equals the partial derivative of the Lagrangian with respect to the associated generalized velocity variable.

A Lagrangian that does not depend on a generalized coordinate q is symmetric to changes in this coordinate, $q \rightarrow q + \delta q$. In the above example, such changes correspond to a rotation about the above-defined axis. This is the classical mechanics analog to the corresponding idea in quantum mechanics that symmetry under a transformation is related to the conservation of a dynamic variable.

1.3 Problems

1. Hoop rolling down a ramp: A hoop of radius R and mass m rolls down a ramp of inclination θ without slipping. Derive the Lagrangian for the hoop and use it to determine the acceleration of the hoop down the ramp. Make a sketch of the setup and the parameters that you use.

2. A mass m with a hole in it slides frictionlessly on a straight wire as shown in figure 1.3. The mass is connected to a wall a distance d from the wire by a spring with spring constant k and the unstretched length of the spring is also d.

   (a) Assuming that gravity does not act, derive the equation for the movement of the mass along the wire using the method of Lagrange.

   (b) Approximate this equation for $x^2 \ll d^2$.

3. A disk of radius a and mass M is rolling around the inside of a cylinder of radius b as shown in figure 1.4.

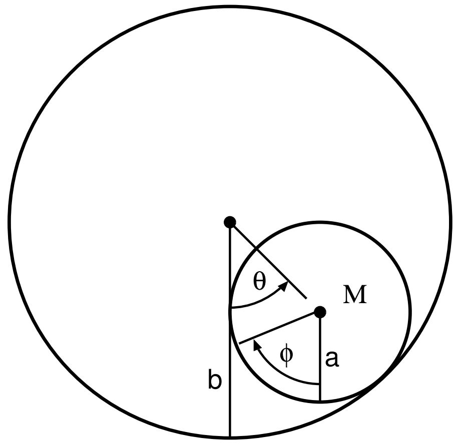

Figure 1.4: Disk of radius a rolling around inside a cylinder of radius b.

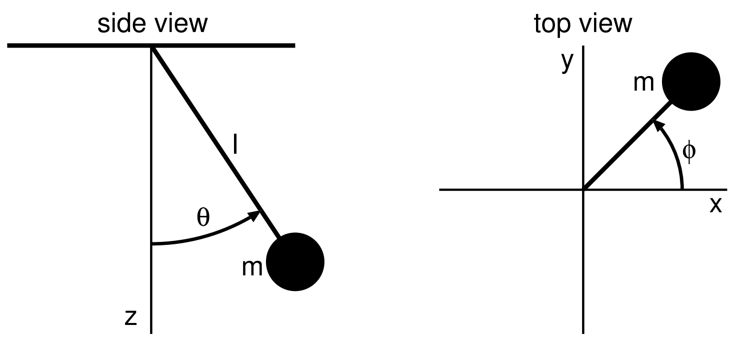

Figure 1.5: Sketch for spherical pendulum.

(a) Assuming the disk rolls without slipping, determine the relationship between $\delta\theta$ and $\delta\phi$.

(b) Assuming that ϕ is defined to be zero when $\theta = 0$, write down the Lagrangian for this system in terms of θ and $\dot{\theta}$.

(c) Derive the governing equation.

4. Spherical pendulum. A mass m which hangs from a string of length l attached to the ceiling can move in both horizontal directions as illustrated in figure 1.5. Its position is specified by the azimuthal angle ϕ and the elevation angle θ.

   (a) Find the Lagrangian for the motion of the mass in terms of θ, ϕ, $\dot{\theta}$, and $\dot{\phi}$.

   (b) Derive the two Lagrange equations for θ and ϕ.

   (c) Note that the Lagrangian does not depend on ϕ. Derive the corresponding conserved generalized momentum. Show that this quantity is the vertical component of the angular momentum of the mass.

   (d) Derive an equation for the angular momentum of the mass in terms of the equilibrium elevation angle θ_0 that occurs when $d^2\theta/dt^2 = 0$. Simplify this expression in the small and large angular momentum limits when

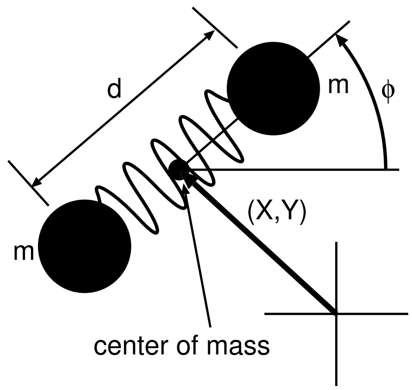

Figure 1.6: Sketch for pucks connected by a spring on an air table.

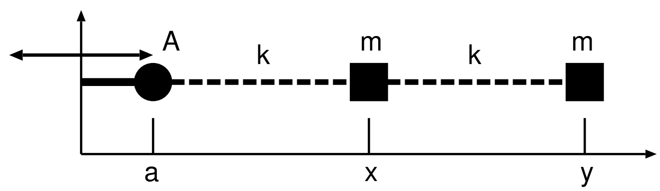

Figure 1.7: Illustration for forced mass-spring system.

i. $|\theta_0| \ll 1$ and

ii. $|\psi_0| = |\pi/2 - \theta_0| \ll 1$.

5. Two pucks of equal mass m are connected on a horizontal air table with a spring of spring constant k and equilibrium length of zero as shown in figure 1.6. Other than the spring constraint, the pucks are assumed to move freely on the air table without friction. The orientation of the pucks is specified by the aximuthal angle ϕ.

   (a) Derive the Lagrangian for this system. Hint: Use the theorem that separates the kinetic energy into a part associated with the motion of the center of mass and a part related to the motions relative to the center of mass. There are 4 degrees of freedom associated with X, Y, d, and ϕ.

   (b) Derive the governing equations for each of these variables. Which of the above degrees of freedom are associated with conserved generalized momenta and what are these quantities?

   (c) Rewrite the d equation eliminating any generalized velocities in favor of the conserved momenta associated with these velocities. Find the equilibrium value of d (i.e., with $d^2d/dt^2 = 0$.)

6. Two masses m and an oscillating support point are connected by two springs with spring constant k and equilibrium length l as shown in figure 1.7. The forcing point moves back and forth according to $a = a_0 \cos(\omega_0 t)$.

   (a) Find the Lagrangian for this system.

(b) Determine the coupled equations of motion for the two masses in terms of their positions x and y.

(c) Find the equilibrium values of x and y with a set to zero, x_0 and y_0, and rewrite the equations in terms of the perturbation lengths $x' = x - x_0$ and $y' = y - y_0$.