# Chapter 1

# Planetary motion

## 1.1 Geometry of the ellipse

The canonical equation for an ellipse with semi-major axis $a$ and semi-minor axis $b$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1.1.1}$$

This is illustrated in figure 1.1.1. The eccentricity of the ellipse is defined

$$\epsilon = c/a \tag{1.1.2}$$

where

$$c^2 = a^2 - b^2 \tag{1.1.3}$$

and $\phi$ is called the true anomaly by astronomers.

We need to derive the polar coordinate $(r, \phi)$ equation for an ellipse with the origin at the left focal point. Noting that $x = r\cos\phi - c$ and $y = r\sin\phi$, equation (1.1.1) becomes

$$r^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) - 2r\left(\frac{c\cos\phi}{a^2}\right) - b^2 = 0 \tag{1.1.4}$$

where we have used equation (1.1.3) to simplify the third term. Solving this quadratic equation for $r$ results in

$$r = \frac{\frac{2c\cos\phi}{a^2} \pm \left[\frac{4c^2\cos^2\phi}{a^4} + \frac{4b^2}{a^2}\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)\right]^{1/2}}{2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)}. \tag{1.1.5}$$

The term in square brackets simplifies to $4/a^2$ where we have used equation (1.1.3) and the identity $\cos^2\phi + \sin^2\phi = 1$. Using this identity in the denominator to eliminate the sine function and rearranging simplifies equation (1.1.5) to

$$r = \frac{b^2}{a}\left(\frac{\epsilon\cos\phi \pm 1}{1 - \epsilon^2\cos^2\phi}\right) = \frac{b^2}{a}\frac{\epsilon\cos\phi \pm 1}{(1 - \epsilon\cos\phi)(1 + \epsilon\cos\phi)}. \tag{1.1.6}$$

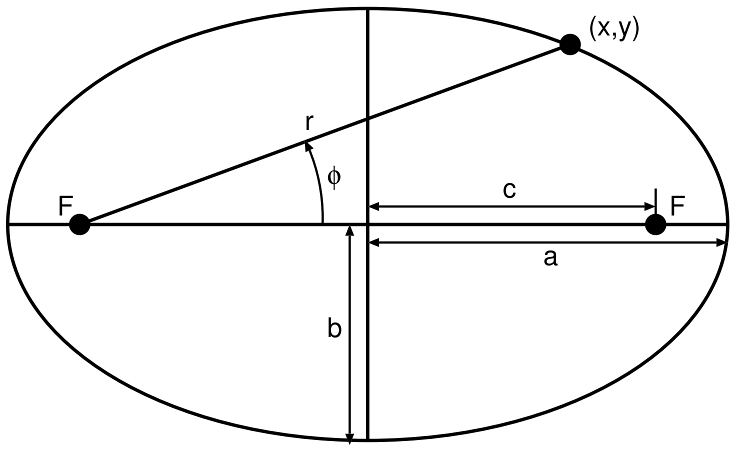


Figure 1.1.1: Definition sketch for an ellipse with semi-major and semi-minor axes $a$ and $b$. The points marked F indicate the focal points of the ellipse.

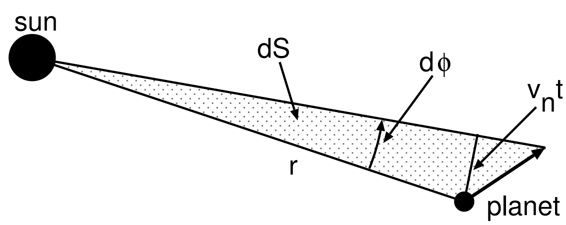


Figure 1.2.1: Sketch illustrating derivation of equal area law.

We now need to choose the sign in the numerator. The negative sign results in negative values for $r$, which is unphysical. Therefore, we choose the plus sign. Cancelling terms in the numerator and denominator and eliminating $b$ in favor of $a$ and $\epsilon$ results finally in

$$r = \frac{a(1-\epsilon^2)}{1-\epsilon\cos\phi}. \tag{1.1.7}$$

The area of an ellipse is given by

$$S = \pi ab. \tag{1.1.8}$$

## 1.2 Kepler's second law

Kepler's second law is really a statement of the conservation of angular momentum. Referring to figure 1.2.1, we note that the area swept out by a planet in its motion around the sun in time $dt$ is

$$dS = \frac{1}{2} r^2 d\phi \tag{1.2.1}$$

where the little triangle on the right end of the area in figure 1.2.1 can be ignored for small $d\phi$. The quantity $v_n$ is the planetary velocity component normal to the radius vector of the planet from the sun. Using $rd\phi = vdt$ and the fact that the angular momentum of the planet in its motion around the sun is $L = mrv_n$ where $m$ is the mass of the planet, equation (1.2.1) can be rewritten

$$\frac{dS}{dt} = \frac{r^2}{2}\frac{d\phi}{dt} = \frac{L}{2m} = constant \tag{1.2.2}$$

due to the conservation of angular momentum. From this we write

$$\frac{d\phi}{dt} = \frac{L}{mr^2} = \frac{C}{r^2} \tag{1.2.3}$$

where $C = L/m$ is the constant orbital angular momentum of the planet per unit mass.

## 1.3 Kepler's first law

Kepler's first law states that planets follow elliptical orbits around the sun with the sun at one focus. Defining $(u, v)$ as the Cartesian components of the planet's velocity in the plane of its orbit, Newton's second law tells us that

$$\frac{d}{dt}(u, v) = -\frac{MG}{r^2}(\cos\phi, \sin\phi) \tag{1.3.1}$$

where $M$ is the mass of the sun and $G$ is the universal gravitational constant. With the help of equation (1.2.3), we find

$$\frac{d}{dt}(u, v) = \frac{d}{d\phi}(u, v)\frac{d\phi}{dt} = \frac{C}{r^2}\frac{d}{d\phi}(u, v) \tag{1.3.2}$$

from which we get

$$\frac{d}{d\phi}(u, v) = -\frac{MG}{C}(\cos\phi, \sin\phi). \tag{1.3.3}$$

This integrates directly to

$$(u, v) = \left(-\frac{MG}{C}\sin\phi + A, \frac{MG}{C}\cos\phi + B\right) \tag{1.3.4}$$

where $A$ and $B$ are arbitrary constants of integration.

We now invoke Sommerfeld's trick to eliminate time dependence and get the radius $r$ solely as a function of the true anomaly $\phi$. We first note that

$$\begin{aligned} u =& \frac{dx}{dt} = \frac{d}{dt}(r\cos\phi) = \frac{dr}{dt}\cos\phi - r\sin\phi\frac{d\phi}{dt} \\ v =& \frac{dy}{dt} = \frac{d}{dt}(r\sin\phi) = \frac{dr}{dt}\sin\phi + r\cos\phi\frac{d\phi}{dt}. \end{aligned} \tag{1.3.5}$$

We eliminate $dr/dt$ from the above equations by computing

$$\begin{aligned} -u\sin\phi + v\cos\phi =& r\sin^2\phi\frac{d\phi}{dt} + r\cos^2\phi\frac{d\phi}{dt} = r\frac{d\phi}{dt} \\ =& \frac{MG}{C}\sin^2\phi - A\sin\phi + \frac{MG}{C}\cos^2\phi + B\cos\phi \end{aligned} \tag{1.3.6}$$

where the second line comes from equation (1.3.4). Simplifying yields

$$r\frac{d\phi}{dt} = \frac{MG}{C} - A\sin\phi + B\cos\phi. \tag{1.3.7}$$

We now invoke equation (2), which gives us

$$r\frac{C}{r^2} = \frac{C}{r} = \frac{MG}{C} - A\sin\phi + B\cos\phi. \tag{1.3.8}$$

Solving for $r$, we arrange the results to allow direct comparison with our equation for an ellipse in polar coordinates (1.1.7):

$$r = \frac{C^2/(MG)}{1 + C(-A\sin\phi + B\cos\phi)/(MG)}. \tag{1.3.9}$$

These equations are consistent if we set

$$A = 0 \qquad B = -\epsilon MG/C \qquad C^2 = a(1-\epsilon^2)MG. \tag{1.3.10}$$

From equation (1.2.3) we therefore relate the planetary orbital angular momentum to the orbital parameters $a$ and $\epsilon$

$$L^2 = m^2[a(1-\epsilon^2)MG] \tag{1.3.11}$$

and compete the proof of Kepler's first law.

## 1.4 Kepler's third law

Kepler's third law states that the square of the period of revolution of a planet is proportional to the cube of the semi-major axis. Integrating equation (1.2.2) over the revolution period $T$ results in

$$\int_0^T \frac{dS}{dt}dt = S = \int_0^T \frac{L}{2m}dt = \frac{LT}{2m} \tag{1.4.1}$$

where $S = \pi ab$ is the area of the ellipse bounded by the planetary orbit. Substituting the area and $L/m$ from equation (1.3.11), squaring, and using the fact that $1-\epsilon^2 = 1 - c^2/a^2 = (a^2-c^2)/a^2 = b^2/a^2$ results in

$$a^3 = \frac{MGT^2}{4\pi^2}, \tag{1.4.2}$$

which proves Kepler's third law.

## 1.5 Solar motion included

The above analysis assumes that the sun is stationary. While not a bad approximation given the relative masses of the Sun and the planets, it is possible to include the sun's motion in

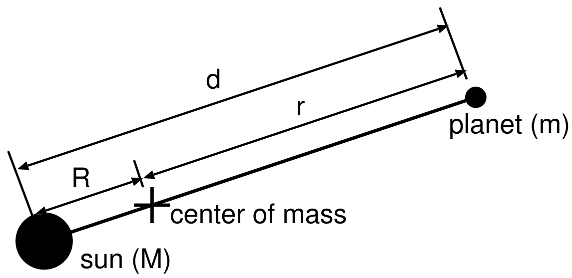


Figure 1.5.1: Sketch for including the motion of the Sun in Kepler problem.

the idealization in which there is only one planet. In this case the sun and the single planet revolve about a common center of mass.

Following Sommerfeld and referring to figure 1.5.1, Newton's third law states that the force of the Sun on the planet is equal and opposite to the force of the planet on the sun,

$$\boldsymbol{F}_S = -\boldsymbol{F}_P = \frac{mMG\hat{\boldsymbol{d}}}{d^2}, \tag{1.5.1}$$

where $\hat{\boldsymbol{d}}$ is a unit vector pointing from the Sun to the planet. From this we see that

$$\boldsymbol{F}_S + \boldsymbol{F}_P = 0, \tag{1.5.2}$$

which shows that the net force on the Sun-planet system is zero, and as a consequence the center of mass of the two systems, as defined in figure 1.5.1 is either stationary or moves with constant velocity, depending on one's (inertial) reference frame.

Using

$$\boldsymbol{F}_S = M\frac{d^2\boldsymbol{x}_S}{dt^2} \qquad \boldsymbol{F}_P = m\frac{d^2\boldsymbol{x}_P}{dt^2} \tag{1.5.3}$$

where $\boldsymbol{x}_S$ and $\boldsymbol{x}_P$ are respectively the positions of the Sun and the planet, we easily show that

$$\frac{\boldsymbol{F}_P}{m} - \frac{\boldsymbol{F}_S}{M} = \frac{d^2\boldsymbol{d}}{dt^2} = -\frac{(m+M)G\hat{\boldsymbol{d}}}{d^2} \tag{1.5.4}$$

where $\boldsymbol{d} = \boldsymbol{x}_P - \boldsymbol{x}_S = d\hat{\boldsymbol{d}}$. The Kepler's first and third law analyses can thus be made to include the motion of the Sun by replacing $M \to m + M$ and $r \to d$ in equation (1.3.1). Kepler's second law remains unchanged.

An alternate way to write equation (1.5.4) is to multiply it by $mM/(m+M)$, resulting in

$$\frac{mM}{m+M}\frac{d^2\boldsymbol{d}}{dt^2} = -\frac{mMG\hat{\boldsymbol{d}}}{d^2} = \boldsymbol{F}_P. \tag{1.5.5}$$

The ratio of masses on the left side can be written

$$\frac{mM}{m+M} = \left(\frac{1}{M} + \frac{1}{m}\right)^{-1} \equiv \mu, \tag{1.5.6}$$

where $\mu$ is called the *reduced mass*. Given this definition, equation (1.5.5) can be written

$$\mu\frac{d^2\boldsymbol{d}}{dt^2} = \boldsymbol{F}_P. \tag{1.5.7}$$