Chapter 1

Hamilton's mechanics

William Rowan Hamilton was an Irish physicist/mathematician from Dublin. Born in 1806, he basically invented modern mechanics in his 60 years and laid the groundwork for the jump to quantum mechanics a hundred years later. Two notable accomplishments bear his name, Hamilton's principle and Hamilton's equations. He also invented quaternians, an extension of complex numbers allowing an alternate representation of rotations in three dimensions, just as complex numbers can be used to represent rotations in two dimensions. Quaternians are closely related to the Pauli spin matrices that are used to represent the spin angular momentum of electrons and other Fermions in quantum mechanics. They are also employed in the software underlying many modern computer games. Finally, Hamilton generalized the Langrangian approach to classical mechanics into the form we use today.

In order to properly appeciate Hamilton's accomplishments, we first study the work of another eminent physicist, Richard Feynman. Feynman invented the method of path integrals, which has become the basis of modern quantum field theory. He popularized the idea of path integrals in his book entitled *QED*. This is an account of the fundamentals of the subject that is very much worth the effort to read. The method of path integrals turns out to be closely related to Fermat's principle, which provides the entry point for our exposition.

1.1 Path integrals

According to Feynman's ideas, the amplitude ψ_i of a matter wave associated with a particle with momentum $\boldsymbol{p}$ and energy E progressing along world line j in spacetime from one eaccording to

$$\psi_j = \exp\left[i\int_j (\boldsymbol{p} \cdot d\boldsymbol{x} - Edt)/\hbar\right].\tag{1.1.1}$$

Notice that $\boldsymbol{p}/\hbar$ is the wave vector $\boldsymbol{k}$ of the matter wave and $E/\hbar$ is just the angular frequency ω. In the case of constant momentum and energy, this equation reduces to

$$\psi_j = \exp\left[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\right],\tag{1.1.2}$$

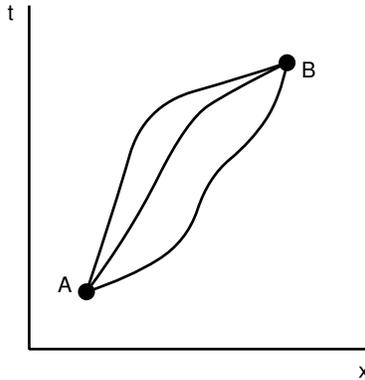which we recognize as the quantum mechanical form of a plane wave.

Figure 1.1.1: Example of multiple possible world lines of particle passing from event A to event B.

The total amplitude for a particle to go from one point in spacetime (i.e., an event) to another is the sum of the amplitudes computed over all possible worldlines connecting these two events:

$$\psi = \sum_j \psi_j. \qquad (1.1.3)$$

Different world lines result in different values of the complex phase of the wave amplitude ψ_j, so a great deal of destructive interference will normally occur in the computation of the total amplitude. However, as we shall see, groups of amplitudes associated with world lines close to certain special world lines will have very close to the same phase, resulting in constructive interference. In the geometric optics limit, these special world lines constitute the classical paths along which the associated particles are observed to travel.

1.2 Constant energy and Fermat's principle

Let us focus temporarily on the case in which the energy E is held constant. In this case the time contribution to the phase $E\Delta t/\hbar$, where Δt is the time difference between the two bounding events, is the same for all world lines. This term therefore disappears in the computation of wave interference and therefore can be omitted from our calculations in this special case. The accumulated phase along the world line (ignoring the constant time contribution) in this case just becomes

$$\psi_j = \int_j \boldsymbol{p} \cdot d\boldsymbol{x}/\hbar = \int_j \boldsymbol{k} \cdot d\boldsymbol{x} = 2\pi \int_j \frac{dx}{\lambda(x)} = 2\pi N_j \qquad (1.2.1)$$

where $\lambda(x)$ is the position-dependent wavelength of the matter wave and N_j is the number of wavelengths exhibited by the matter wave along the world line between the two bounding events.

Let us first assume that the momentum is also constant as the particle moves from point A to point B (the space coordinates of the two bounding events) with possible alternate paths,
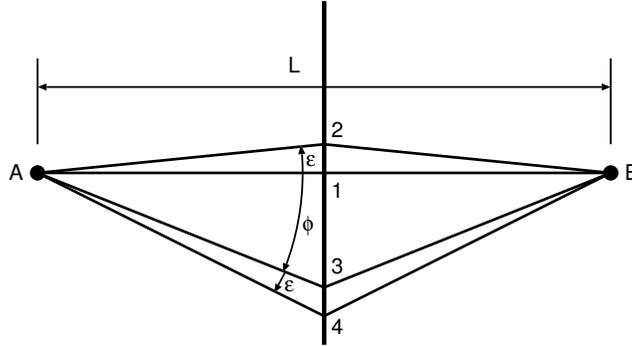
Figure 1.2.1: Alternate trajectories for a particle to move from point A to point B.

as illustrated in figure 1.2.1. The distances travelled along each of the paths are

$$
\begin{aligned}
D_1 &= L \\
D_2 &= L/\cos(\varepsilon) \\
D_3 &= L/\cos(\phi) \\
D_4 &= L/\cos(\phi + \varepsilon) = L/\left[\cos(\phi)\cos(\varepsilon) - \sin(\phi)\sin(\varepsilon)\right].
\end{aligned}
\tag{1.2.2}
$$

For $|\varepsilon| \ll 1$, the difference in the number of wavelengths traversed along paths 1 and 2 may be approximated to the lowest non-zero order in ε as

$$
\Delta N_{21} = \frac{D_2 - D_1}{\lambda} = \frac{\varepsilon^2 L}{2\lambda}
\tag{1.2.3}
$$

while the difference in the number of wavelengths between paths 3 and 4 is approximately

$$
\Delta N_{43} = \frac{D_4 - D_3}{\lambda} = \frac{\varepsilon L \tan(\phi)}{\lambda \cos(\phi)}.
\tag{1.2.4}
$$

Constructive interference between paths 1 and 2 requires that $|\Delta N_{21}| \ll 1$, or

$$
|\varepsilon| \ll \left(\frac{\lambda}{L}\right)^{1/2}
\tag{1.2.5}
$$

while constructive interference between paths 3 and 4 requires that $|\Delta N_{34}| \ll 1$, or

$$
|\epsilon| \ll \frac{\lambda}{L}.
\tag{1.2.6}
$$

The factor of 2 in equation (1.2.3) and the trig functions in equation (1.2.4) have been dropped as they are both of order unity. Since we assume that each world line traverses many wavelengths, $\lambda/L \ll 1$. As a consequence, $\lambda/L \ll (\lambda/L)^{1/2}$, and the range of angles over which constructive interference can occur for the 1-2 case is much greater than for the 3-4 case. From this we conclude that the straight line (path 1) is a special world line along which particles travel in the classical limit when the momentum is fixed, whereas the crooked line (path 3) is not. This is consistent with Newton's first law.

Figure 1.2.2: Dividing a line segments up into sub-segments.

The alternate paths in figure 1.2.1 represent only a small selection of the possible paths between points A and B. However, one could divide the actual path between these points into a set of sub-paths and apply the above reasoning to each sub-path, resulting in the same conclusion; if the momentum is constant, the classical path is a straight line. Letting the number of segments increase indefinitely generalizes the conclusion to include all possible paths between points A and B.

Note that the path with the broadest spread of neighboring paths yielding the strongest constructive interference is the one with the least number of wavelengths between the two points. This is not a coincidence. If the momentum is not constant, i.e., if the particle is subject to a conservative force or if a dielectric medium exists in the case of light, then the path with the smallest number of wavelengths is no longer a straight line. However the bundle of paths close to this path will still exhibit the strongest constructive interference and will therefore be the preferred path for the associated particles in the geometrical optics limit. The key point is that the path with an extremum in the number of wavelengths (not necessarily a minimum) is the path with the largest bundle of nearby paths exhibiting constructive interference.

For light, the path with the fewest number of wavelengths is also the path traversed by the light in the least time, at least as long as the phase speed of the light u_p equals its group velocity u_g. In this case

$$\Delta t = \frac{\Delta x}{u_g} = \frac{\Delta x}{u_p} = \frac{k \Delta x}{\omega} = \frac{2\pi \Delta x}{\omega \lambda} = \frac{2\pi}{\omega} N \tag{1.2.7}$$

where N is the number of wavelengths traversed. Thus, the principle of minimum number of wavelengths traversed is equivalent to Fermat's principle of least time. Note that if the index of refraction depends on wavenumber, so that the dispersion relation for light in a dielectric is dispersive, Fermat's principle is not strictly true if "minimum time" means the minimum time for a wave packet to traverse the path. (Minimum time in the context of Fermat's principle is defined in terms of the time calculated from the phase speed rather than the group velocity.)

1.3   Hamilton's principle

Let us return to the more general case in which the total energy may vary. Recognizing $d\boldsymbol{x}/dt \equiv \boldsymbol{v}$ as the velocity of the particle, equation (1.1.1) can be rewritten as

$$\psi_j = \exp\left[i \int_j (\boldsymbol{p} \cdot \boldsymbol{v} - E) dt / \hbar\right]. \tag{1.3.1}$$

In the case in which we are dealing with a single non-relativistic particle with momentum $\boldsymbol{p} = m\boldsymbol{v}$ and total energy equal to the sum of the particle's kinetic and potential energies

$E = T + V$, the integrand in equation (1.3.1) equals the Lagrangian L:

$$\boldsymbol{p} \cdot \boldsymbol{v} - E = mv^2 - mv^2/2 - V = mv^2/2 - V = L. \tag{1.3.2}$$

Thus the wave amplitude for the jth world line between the initial and final events becomes

$$\psi_j = \exp\left(i \int_j Ldt/\hbar\right). \tag{1.3.3}$$

Generalizing the result from the previous section, the classical particle world line will be that which yields an extremum for the value of the integral inside the exponential function

$$S \equiv \int_j Ldt. \tag{1.3.4}$$

This is called *Hamilton's principle* and S is called the *action*.

The above analysis applies to a single particle. For multiple particles (ignoring the issues posed by identical particles), the total amplitude is the product of the single particle amplitudes ψ_j^k:

$$\Psi_j = \psi_j^1 \cdot \psi_j^2 \cdot \psi_j^3 ... = \exp\left[i \int_j (L_1 + L_2 + L_3 + ...)dt/\hbar\right] = \exp\left(i \int_j Ldt/\hbar\right). \tag{1.3.5}$$

The L_k are the Lagrangians for the individual particles and $L(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2, ..., t)$ is the sum of the individual Lagrangians. In systems with constraints, the total Langrangian may be rewritten in terms of appropriate generalized coordinates and velocities $L(q_i, \dot{q}_i, t)$ as we have done previously.

1.4 Lagrange equations from Hamilton's principle

We now use the calculus of variations to see if Hamilton's principle implies the Lagrange equations. If it does, then the circle is closed and we know that the geometrical optics limit of quantum mechanics is equivalent to classical mechanics.

We now write Hamilton's principle in terms of a Lagrangian that is a function of generalized coordinates and velocities,

$$\delta \int_A^B L(q_i, \dot{q}_i, t)dt = 0, \tag{1.4.1}$$

where A and B are the fixed end points and δ indicates the difference between integrals taken over slightly different world lines. The δ acts like any other differential operator, resulting in

$$\delta \int_A^B L(q, \dot{q}, t)dt = \sum_i \int_A^B \left(\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i\right) dt = 0 \tag{1.4.2}$$

where the differencing is taken at fixed time. The quantity $\delta\dot{q}_i = \dot{q}_i^\alpha - \dot{q}_i^\beta$, where the superscript indicates neighboring paths, may be written

$$\delta\dot{q}_i = \frac{dq_i^\alpha}{dt} - \frac{dq_i^\beta}{dt} = \frac{d}{dt}(q_i^\alpha - q_i^\beta) = \frac{d}{dt}\delta q_i, \qquad (1.4.3)$$

which shows that the total time derivative and the differencing commute. Integrating by parts, we find that

$$\int_A^B \frac{\partial L}{\partial\dot{q}_i}\frac{d}{dt}\delta q_i dt = \left(\frac{\partial L}{\partial\dot{q}_i}\delta q_i\right)_A^B - \int_A^B \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\delta q_i dt. \qquad (1.4.4)$$

The first term on the right side of equation (1.4.4) is zero because by hypothesis $\delta q_i = 0$ at the end points. Substituting the second term into equation (1.4.2) results in

$$\sum_i \int_A^B \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \frac{\partial L}{\partial q_i}\right]\delta q_i dt = 0. \qquad (1.4.5)$$

Since the δq_i are independent and of arbitrary form, the quantity inside the square brackets must be zero to satisfy this equation, resulting in the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \qquad (1.4.6)$$

1.5 Hamilton's equations

An alternate approach to mechanics developed by Hamilton emphasizes the generalized momenta $p_i = \partial L/\partial q_i$ over the generalized velocities $\dot{q}_i$. In this approach Hamilton defined a new function, now called the *Hamiltonian*, as

$$H(\Pi_j, q_j, \dot{q}_j, t) = \sum_i \Pi_i \dot{q}_i - L(q_j, \dot{q}_j, t) \qquad (1.5.1)$$

where for now we leave Π_i unspecified.

It is clear that

$$\frac{\partial H}{\partial \Pi_j} = \dot{q}_j. \qquad (1.5.2)$$

Furthermore,

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_j}\right) = -\dot{p}_j, \qquad (1.5.3)$$

where Lagrange's equation has been used to relate $\partial L/\partial q_j$ to $\dot{p}_j$. Finally,

$$\frac{\partial H}{\partial\dot{q}_j} = \Pi_j - \frac{\partial L}{\partial\dot{q}_j} = \Pi_j - p_j. \qquad (1.5.4)$$

We now set $\Pi_j = p_j$, which has two results. First, equations (1.5.2) and (1.5.3) become what are called *Hamilton's equations*

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \qquad \frac{\partial H}{\partial q_j} = -\dot{p}_j. \qquad (1.5.5)$$

Second, equation (1.5.4) shows us that

$$\frac{\partial H}{\partial \dot{q}_j} = 0, \qquad (1.5.6)$$

i.e., the Hamiltonian is *not* a function of $\dot{q}_j$, but only of p_j and q_j, and possibly of time t:

$$H = H(p_j, q_j, t). \qquad (1.5.7)$$

Hamilton's equations constitute an alternative expression of Newton's second law. In addition, the Hamiltonian is used extensively in most formulations of quantum mechanics. This is the primary reason we introduce it here.

Since the Hamiltonian in most cases is just the total energy, we can just write the energy in terms of the generalized positions and momenta to get the Hamiltonian. However, the safe way to do this comes from equation (1.5.1)

$$H = \sum_i p_i \dot{q}_i - L \qquad (1.5.8)$$

with $\dot{q}_j$ eliminated in favor of p_j. The Hamiltonian equals the total energy when

$$\sum_i p_i \dot{q}_i = 2T, \qquad (1.5.9)$$

so that we have

$$H = 2T - L = 2T - T + V = T + V = E. \qquad (1.5.10)$$

Equation (1.5.8) isn't satisfied in odd situations in which the system under consideration is not isolated.

The Hamiltonian has another interesting property

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right) = \frac{\partial H}{\partial t} \qquad (1.5.11)$$

where the last step invokes Hamilton's equations (1.5.5). Thus, if the Hamiltonian has no explicit dependence on time, it is conserved in real dynamical processes. This is true whether or not the Hamiltonian equals the total energy. Thus, there are four possibilities:

- The sum of the generalized momenta times the corresponding generalized velocities equals twice the kinetic energy and the Hamiltonian is time independent: The Hamiltonian equals the energy and is conserved.

- The sum of the generalized momenta times the corresponding generalized velocities equals twice the kinetic energy but the Hamiltonian is time dependent: The Hamiltonian equals the energy but is not conserved.

- The sum of the generalized momenta times the corresponding generalized velocities does not equal twice the kinetic energy but the Hamiltonian is time independent: The Hamiltonian does not equal the energy but is conserved.

- The sum of the generalized momenta times the corresponding generalized velocities does not equal twice the kinetic energy and the Hamiltonian is time dependent: The Hamiltonian does not equal the energy and is not conserved.

1.6 Electromagnetism

Adding electromagnetism to Lagrangian and Hamiltonian mechanics requires an extension of the theory because the magnetic force on a charged particle is not derivable from a potential and also depends on the velocity of the particle. In physics 222 we introduced the concept of potential momentum $\boldsymbol{Q}$ as well as the kinetic and total momenta, $\boldsymbol{p}$ and $\boldsymbol{\Pi}$, related in the following way:

$$\boldsymbol{\Pi} = \boldsymbol{p} + \boldsymbol{Q}. \tag{1.6.1}$$

In advanced mechanics the total momentum is called the *canonical* momentum and the kinetic momentum is just called the "momentum" or the "ordinary momentum". The potential momentum doesn't have a special name, but in the case of electromagnetism it is just $q\boldsymbol{A}$ where q is the charge on the particle of interest and $\boldsymbol{A}$ is the vector potential.

The canonical momentum is related to the wave vector of the matter wave associated with the particle by the de Broglie relation

$$\boldsymbol{\Pi} = \hbar\boldsymbol{k} \tag{1.6.2}$$

whereas the ordinary momentum is related to the particle mass and velocity in the usual way. In the non-relativistic case:

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{1.6.3}$$

Thus, the total or canonical momentum may be written

$$\boldsymbol{\Pi} = m\boldsymbol{v} + q\boldsymbol{A}. \tag{1.6.4}$$

The total energy is written similarly as a sum of kinetic and potential parts

$$E = \frac{1}{2}mv^2 + q\phi \tag{1.6.5}$$

where ϕ is the electrostatic potential. The total energy is related to the matter wave frequency by the Planck relation $E = \hbar\omega$ while the kinetic energy is related to the velocity.

Looking back on the derivation of Hamilton's principle from Feynman's path integral formulation of quantum mechanics, it is clear that the momentum in equation (1.3.2) is the total

or canonical momentum since it originates as the wave vector times $\hbar$. Thus, the Lagrangian for a single charged particle in the presence of electric and magnetic fields becomes

$$
\begin{aligned}
L &= \mathbf{\Pi} \cdot \boldsymbol{v} - E \\
&= (m\boldsymbol{v} + q\boldsymbol{A}) \cdot \boldsymbol{v} - \frac{1}{2}mv^2 - q\phi \\
&= \frac{1}{2}mv^2 - q(\phi - \boldsymbol{v} \cdot \boldsymbol{A})
\end{aligned}
\tag{1.6.6}
$$

where ϕ is the electromagnetic scalar potential. The components of the generalized momentum are obtained from the Lagrangian

$$
\left(\frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y}, \frac{\partial L}{\partial v_z} \right) = m\boldsymbol{v} + \boldsymbol{A} = \mathbf{\Pi}
\tag{1.6.7}
$$

as expected, verifying that the generalized momentum in this case is actually the canonical momentum. Lagrange's equations can be written in vector form as

$$
\frac{d\mathbf{\Pi}}{dt} - \boldsymbol{\nabla}L = \frac{d\mathbf{\Pi}}{dt} + q\boldsymbol{\nabla}(\phi - \boldsymbol{v} \cdot \boldsymbol{A}),
\tag{1.6.8}
$$

or using the definition of $\mathbf{\Pi}$ in equation (1.6.7), as

$$
m\frac{d\boldsymbol{v}}{dt} = -q\frac{d\boldsymbol{A}}{dt} - q\boldsymbol{\nabla}(\phi - \boldsymbol{v} \cdot \boldsymbol{A}).
\tag{1.6.9}
$$

The total time derivative of the vector potential is the time rate of change of the potential at the location of the charged particle on which it is acting. Using the chain rule, it may be written

$$
\frac{d\boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{A}
\tag{1.6.10}
$$

where we recognize $d\boldsymbol{x}/dt = \boldsymbol{v}$ since $\boldsymbol{x}(t)$ is the position of the particle. Equation (1.6.9) thus becomes

$$
m\frac{d\boldsymbol{v}}{dt} = q\left[-\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \right] + q\left[-\boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{A} + \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) \right].
\tag{1.6.11}
$$

We recognize the quantity inside the first set of square brackets as the electric field $\boldsymbol{E}$. Application of a vector identity shows that the quantity inside the second set of brackets is $\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$. Noting that $\boldsymbol{\nabla} \times \boldsymbol{A}$ is just the magnetic field $\boldsymbol{B}$, equation (1.6.11) finally becomes

$$
m\frac{d\boldsymbol{v}}{dt} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B},
\tag{1.6.12}
$$

which is the Lorentz force law of electromagnetism. This verifies that the assumption that $q\boldsymbol{A}$ and $q\phi$ represent the potential momentum and the potential energy of a particle with charge q in electric and magnetic fields is equivalent to the results of classical electromagnetism.
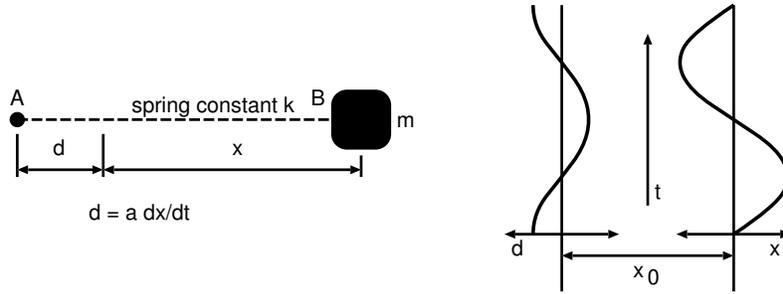
Figure 1.8.1: Sketch of active oscillator.

1.7 Relativity and Hamilton's principle

Hamilton's principle is a perfectly relativistic concept. Writing equation (1.3.4) in the form

$$\delta \int L dt = \delta \int (\boldsymbol{p} \cdot \boldsymbol{v} - E) dt = 0, \qquad (1.7.1)$$

(so that the Hamiltonian is the energy) we note that the Lagrangian is the dot product of two 4-vectors divided by γ:

$$L = \boldsymbol{p} \cdot \boldsymbol{v} - E = \underline{p} \cdot \underline{u}/\gamma \qquad (1.7.2)$$

where $\underline{p} = (\boldsymbol{p}, E/c)$ is the 4-momentum and $\underline{u} = (\gamma\boldsymbol{v}, \gamma c)$ is the 4-velocity. Recall that the 4-momentum for a free particle is just the mass m of the object times the 4-velocity $\underline{p} = m\underline{u}$, so $\underline{p} \cdot \underline{u} = m\underline{u} \cdot \underline{u} = -mc^2$. The relativistic free particle Lagrangian is therefore

$$L = -\frac{mc^2}{\gamma} = -mc^2[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2]^{1/2}. \qquad (1.7.3)$$

Hamilton's principle may be expressed alternately as an integral over proper time. This is because $dt/\gamma = d\tau$ where $d\tau$ is the increment of proper time along the world line of the particle under consideration. Thus, Hamilton's principle for a free particle in the relativistic case can be written

$$\delta \int L dt = -\delta \int \left(mc^2/\gamma \right) dt = -mc^2 \delta \int d\tau = 0. \qquad (1.7.4)$$

Hence the world line traversed by the free particle is one with an extremum in the elapsed proper time. A world line with constant velocity leading directly from the starting point to the end point satisfies this criterion as expected.

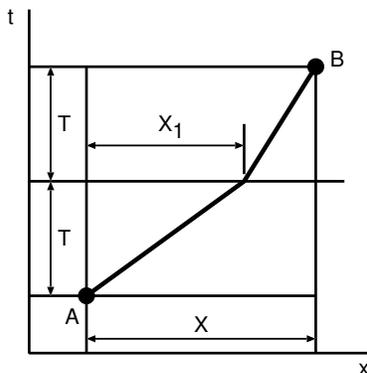## 1.8 Problems

1. Hamiltonian method test run:

Figure 1.8.2: Illustration of world line to be varied.

(a) Derive the Lagrangian for a mass-spring system with mass m and spring constant k in terms of x and $\dot{x}$ where x is the deviation of the mass from its equilibrium position.

(b) Compute the generalized momentum of the mass associated with the variable x.

(c) Derive the Hamiltonian for this system using equation (1.5.1). Be sure to eliminate the generalized velocity in favor of the generalized momentum.

(d) Derive the Hamiltonian governing equations for this case.

(e) Solve Hamilton's equations for $x(t)$ and $p_x(t)$.

2. A mass m slides horizontally on a frictionless table constrained by a string passing through a hole in the table that can be pulled or released arbitrarily, resulting in a string length from the hole to the mass of $r(t)$. The mass is also moving around the hole with azimuthal angle ϕ.

(a) Derive the Lagrangian for the motion of the mass.

(b) Compute the generalized momentum associated with ϕ. Is it conserved?

(c) Derive the Hamiltonian of the system.

(d) Solve for $\dot{\phi}$ in terms of p_ϕ. Hint: Do you even have to derive the governing equations from the Hamiltonian in this case?

(e) Is the energy of the system conserved? Explain.

3. Active harmonic oscillator: Figure 1.8.1 illustrates an actively controlled mass-spring system in which the left end of the spring (point A) is moved in lockstep with the motion of the mass such that the displacement d of A to the left is somehow made proportional to the time rate of change of the right end of the spring (point B)

$$d = a\dot{x}$$

via some servomechanism, where a is a constant.

(a) Realizing that the net extension of the spring from its equilibrium length is $x + d$, write down the Lagrangian for the mass-spring system. Is the Lagrangian quadratic in $\dot{x}$?

(b) Derive the governing equation and determine the general solution $x(t)$.

(c) Derive the generalized x momentum p_x and write the Hamiltonian in terms of p_x and x. (Note: This expression can be greatly simplified into one term containing $(p_x + kax)$ plus another term $kx^2/2$. Carry out this simplification.)

(d) Is the Hamiltonian conserved?

(e) Is the Hamiltonian equal to the total energy? If not, explain the process by which energy is transferred in and out of the mass-spring system. Hint: Re-express the Hamiltonian in terms of $\dot{x}$ and x and compare with the potential and kinetic energies used to derive the Lagrangian.

4. Using Lagrange's method, determine the conserved generalized momenta for a relativistic free particle, starting with equation (1.7.3).

5. Relativistic electromagnetic force.

(a) By appropriate modification of equation (1.6.6) show that the relativistic Lagrangian with electromagnetism is

$$L = -mc^2[1 - (v_x^2 + v_y^2 + v_z^2)/c^2]^{1/2} - q(v_x A_x + v_y A_y + v_z A_z - \phi).$$

Hint: Seek help in section 1.7.

(b) Compute the generalized momenta for the variables x, y, and z.

(c) Derive the governing equations using the method of Lagrange. Compare the results with equation (1.6.9).

6. Evaluate the elapsed proper time from A to B in figure 1.8.2 as a function of X_1. Assume that $c = 1$ in this calculation. Find the value of X_1 for which the elapsed proper time is an extremum. Determine whether this extremum is a minimum or a maximum.