

Chapter 1

D'Alembert's principle and applications

1.1 D'Alembert's principle

The principle of virtual work states that the sum of the incremental virtual works done by all external forces \mathbf{F}_i acting in conjunction with virtual displacements $\delta\mathbf{s}_i$ of the point on which the associated force is acting is zero:

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta\mathbf{s}_i = 0. \quad (1.1.1)$$

This technique is useful for solving statics problems, with static forces of constraint. A static force of constraint is one that does no work on the system of interest, but merely holds a certain part of the system in place.

In a statics problem there are no accelerations. We can extend the principle of virtual work to dynamics problems, i.e., ones in which real motions and accelerations occur, by introducing the concept of inertial forces. For each parcel of matter in the system with mass m , Newton's second law states that

$$\mathbf{F} = m\mathbf{a}. \quad (1.1.2)$$

We can make this dynamics problem look like a statics problem by defining an *inertial force*

$$\mathbf{F}^* = -m\mathbf{a} \quad (1.1.3)$$

and rewriting equation (1.1.2) as

$$\mathbf{F}_{total} = \mathbf{F} + \mathbf{F}^* = 0. \quad (1.1.4)$$

D'Alembert's principle is just the principle of virtual work with the inertial forces added to the list of forces that do work:

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta\mathbf{s}_i + \sum_j \mathbf{F}_j^* \cdot \delta\mathbf{s}_j = 0. \quad (1.1.5)$$

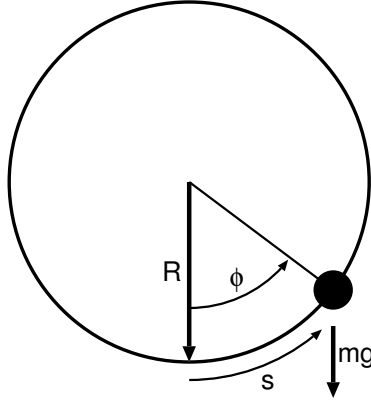


Figure 1.1.1: Sketch of bead of mass m sliding frictionlessly on a vertical hoop of radius R under the influence of gravity.

1.1.1 Mass falling under gravity

A trivial example would be a mass m falling under the effect of a constant gravitational field g . With z positive upward, the force on the mass is $-mg$ and the work due to this force under vertical displacement δz is $\delta W_G = -mg\delta z$. The inertial force is $-m(d^2z/dt^2)$ and the work is $\delta W_I = -m(d^2z/dt^2)\delta z$. Setting the sum of the two to zero gives us

$$-mg\delta z - m(d^2z/dt^2)\delta z = -[mg + m(d^2z/dt^2)]\delta z = 0, \quad (1.1.6)$$

from which we infer the expected result

$$\frac{d^2z}{dt^2} = -g. \quad (1.1.7)$$

D'Alembert's principle offers no advantages over normal procedures in this case. However, it becomes more economical in problems with constraints.

1.1.2 Bead on frictionless vertical hoop

Figure 1.1.1 illustrates the slightly more interesting problem of a bead sliding frictionlessly around a vertically oriented hoop of wire. Here the force of gravity is not in the direction of motion. The component of gravity normal to the hoop does no work on the bead. Nor does the force of the hoop on the bead that constrains the bead to move in a circle. The work on the bead due to gravity for a small displacement $\delta s = R\delta\phi$ along the wire is $\delta W_G = -mgR\sin\phi\delta\phi$. The acceleration of the bead also has two components, a radial component $a_r = -v^2/R$, where the tangential velocity is $v = ds/dt = R(d\phi/dt)$, and a tangential component $a_t = d^2s/dt^2 = R(d^2\phi/dt^2)$. The radial component of the inertial force mv^2/R does no work. However, the tangential component $-ma_t = -mR(d^2\phi/dt^2)$ does: $\delta W_I = -mR(d^2\phi/dt^2)\delta s = -mR^2(d^2\phi/dt^2)\delta\phi$. D'Alembert's principle thus gives us

$$[-mgR\sin\phi - mR^2(d^2\phi/dt^2)]\delta\phi = 0 \quad (1.1.8)$$

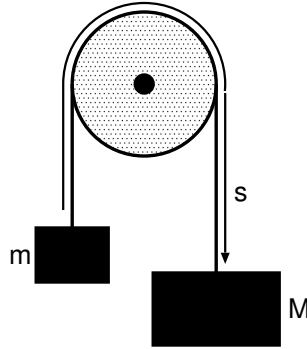


Figure 1.1.2: Sketch of Atwood's machine.

from which we get the governing equation

$$\frac{d^2\phi}{dt^2} + \frac{g \sin \phi}{R} = 0. \quad (1.1.9)$$

(This could equally well have been expressed in terms of s rather than ϕ .)

1.1.3 Atwood's machine

D'Alembert's principle makes the venerable Atwood's machine, illustrated in figure 1.1.2, trivial to analyze. The state of the machine is determined by the positions of the two masses along the U-shaped coordinate s looping over the frictionless pulley with the string. The work done by gravity on the left-hand mass under the displacement δs is $\delta W_L = -mg\delta s$, while the gravity acting on the right-hand mass produces work $\delta W_R = +Mg\delta s$. The inertial work on the two masses gives $\delta W_I = -(M + m)(d^2s/dt^2)\delta s$, resulting in

$$\delta W = [(M - m)g - (M + m)(d^2s/dt^2)] \delta s = 0 \quad (1.1.10)$$

or

$$\frac{d^2s}{dt^2} = g \frac{M - m}{M + m}. \quad (1.1.11)$$

Note that the force of constraint by the pulley (which is assumed to be free to rotate but held rigidly in place and with negligible moment of inertia) does not enter the problem at all.

1.2 Distributed forces

So far we have dealt with a limited number of forces that each act at a point on the system of interest. However, forces like gravity as well as inertial forces act on each mass point in the system, making the calculation of the virtual work more complex.

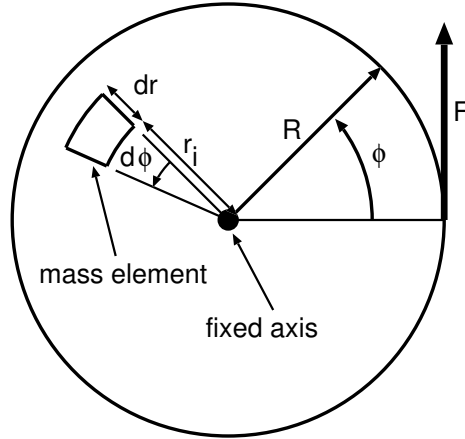


Figure 1.2.1: Sketch for rotating disk with angular acceleration.

1.2.1 Falling mass reconsidered

Let's return to the simple case of a mass falling in a uniform gravitational field. If the mass is extended but not rotating, the vertical displacement of each mass element Δm_i is the same: δz . The work due to gravity on each point is therefore the same: $\delta W_{Gi} = -\Delta m_i g \delta z$. In this case the work due to the inertial force associated with the vertical acceleration is also simple: $\delta W_{Ii} = -\Delta m_i (d^2 z / dt^2) \delta z$ where we have recognized that the accelerations associated with each point $d^2 z_i / dt^2$ are all the same as that of some reference point z fixed to the object. There are also forces exerted on each mass element by neighboring mass elements that do work. However, as we now sum over all mass elements, these internal forces cancel by virtue of Newton's third law. Performing this sum, we find that

$$\delta W = - (g + d^2 z / dt^2) \sum_i \Delta m_i \delta z = - (g + d^2 z / dt^2) m \delta z, \quad (1.2.1)$$

where m is the total mass of the object. This gives rise to the expected result

$$\frac{d^2 z}{dt^2} = -g. \quad (1.2.2)$$

1.2.2 Rotating disk and moment of inertia

Figure 1.2.1 shows a rotating disk with angular acceleration induced by a force F applied on the right side of the disk. The work done by the force with an angular displacement $\delta \phi$ is $\delta W_F = RF \delta \phi$. the work done by the inertial force on mass element dm_i is $\delta W_{Ii} = -dm_i r_i (d^2 \phi / dt^2) r_i \delta \phi$ where r_i is the radius at which the mass element is located. Ignoring internal forces between mass elements as before, the total virtual work is therefore

$$\delta W = \left[RF - \sum_i dm_i r_i^2 (d^2 \phi / dt^2) \right] \delta \phi \quad (1.2.3)$$

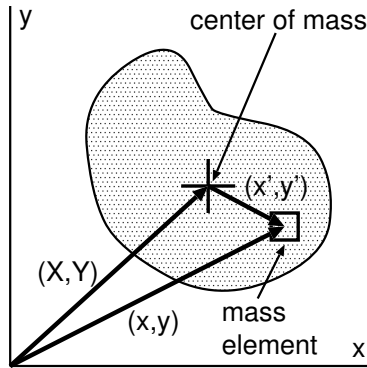


Figure 1.2.2: Sketch for parallel axis theorem. The fixed point denoted by the cross becomes the center of mass.

and the governing equation is

$$I \frac{d^2 \phi}{dt^2} = RF \quad (1.2.4)$$

where the *moment of inertia* is

$$I = \sum_i dm_i r_i^2. \quad (1.2.5)$$

Using the methods of calculus, we write $dm_i = \rho r_i dr_i d\phi dz$ where the (constant) mass density is ρ and the variable z is normal to the page. Upon letting the differentials become infinitesimal, I becomes

$$I = \rho \int_0^w \int_0^{2\pi} \int_0^R r^3 dr d\phi dz = \frac{\pi w R^4 \rho}{2}, \quad (1.2.6)$$

where w is the thickness of the disk. Recognizing that the mass of the disk is $M = \pi R^2 w \rho$, elimination of the density results in

$$I = \frac{MR^2}{2}. \quad (1.2.7)$$

The moments of inertia for other common shapes may be obtained by similar methods.

1.2.3 Parallel axis theorem

In the above case the moment of inertia was calculated about the center of mass of the disk. Sometimes we need to make this calculation for rotations about some other point. The *parallel axis theorem* shows how to do this once the moment of inertia about the center of mass has been calculated.

Referring to figure 1.2.2, the moment of inertia about the origin of the illustrated mass, which is in the form of an irregularly shaped plate of thickness h normal to the page, is

$$I = h \int r^2 dm = h \int \int r^2 \rho dx dy = h \int \int (x^2 + y^2) \rho dx dy, \quad (1.2.8)$$

where in general the density $\rho(x, y)$ may depend on position in the plane and where the bounds on the integral extend over the mass. Let us substitute $(x, y) = (X, Y) + (x', y')$ where (X, Y) points to a fixed location. The above integral then becomes

$$\begin{aligned} I &= (X^2 + Y^2)h \iint \rho dx dy \\ &+ 2h \iint (Xx' + Yy')\rho dx' dy' \\ &+ h \iint (x'^2 + y'^2)\rho dx' dy'. \end{aligned} \quad (1.2.9)$$

(The change of variables in the integral is trivial, as $dx = dx'$ and $dy = dy'$.) The first term on the right side of the equals sign is just the mass M of the object times the square of the distance from the origin to the fixed point $R^2 = X^2 + Y^2$. The third term is the moment of inertia about the fixed point. Anticipating that this point will become the center of mass, we denote the moment of inertia about this point I_{CM} .

The second term can be written in the form of a dot product

$$2(X, Y) \cdot h \iint (x', y')\rho dx' dy' \quad (1.2.10)$$

which must be zero for all choices of the original origin if (X, Y) is to become the location of the center of mass. To see why this is so, substitute $(x', y') = (x, y) - (X, Y)$ in the integral in equation (1.2.10) and change from primed to unprimed integrand variables:

$$h \iint (x, y)\rho dx dy - (X, Y)h \iint \rho dx dy = 0. \quad (1.2.11)$$

The coefficient of (X, Y) is just the mass of the object, whereas the first term is just the mass times mass-weighted average of (x, y) . This is just the mass times the center of mass, and therefore

$$(X, Y) = \frac{h}{M} \iint (x, y)\rho dx dy \quad (1.2.12)$$

is just the position of the center of mass. Equation (1.2.9) becomes

$$I = MR^2 + I_{CM} \quad (1.2.13)$$

a statement of the parallel axis theorem: the moment of inertia about an arbitrary axis (for a rotation axis normal to the x - y plane) is just the moment of inertia for rotations about the center of mass (also normal to the x - y plane) plus the mass times the square of the distance between the arbitrary axis and the center of mass.

1.2.4 Disk rolling down ramp

The problem of a disk rolling down a ramp under the influence of gravity can be solved economically using the parallel axis theorem. Two real forces act on the disk, gravity and

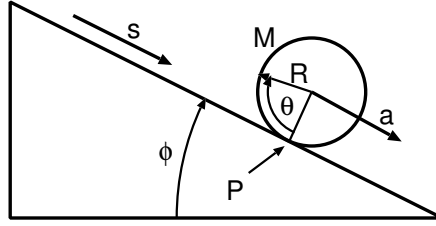


Figure 1.2.3: Sketch of a disk rolling down a ramp.

the force (with normal and tangential components) of the ramp on the disk. Since in rolling motion the point P on the disk in contact with the ramp (see figure 1.2.3) is not moving, the force of the ramp on the disk is one of constraint and does no work. Since the point P is stationary, the motion of the disk over a short interval can be expressed as a rotation about the point P . The work done by the gravitational force as the disk moves a distance δs down the ramp is $\delta W_G = Mg \sin \phi \delta s = MgR \sin \phi \delta \theta$ and the work due to the integral of the inertial force acting on all pieces of the disk is $\delta W_I = -I(d^2\theta/dt^2)\delta\theta$ in analogy with the disk in section 1.2.2, where the moment of inertia is that appropriate for rotation about the point P in figure 1.2.3. Using the parallel axis theorem, we have

$$I = MR^2 + \frac{MR^2}{2} \quad (1.2.14)$$

and hence

$$\delta W = \left(MgR \sin \phi - \frac{3MR^2}{2} \frac{d^2\theta}{dt^2} \right) \delta\theta = 0. \quad (1.2.15)$$

Noting that $ds = R d\theta$, this leads to the governing equation

$$\frac{d^2s}{dt^2} = \frac{2g \sin \phi}{3}. \quad (1.2.16)$$

Thus, the acceleration down the ramp in rolling motion is less than would be expected from frictionless sliding, $g \sin \phi$.

1.2.5 Motion of solid bodies in two dimensions

The inertial virtual work in general for a solid body in two-dimensional rotation and translation can be written as the sum of the works for each mass element of the solid:

$$\delta W_I = - \sum_i \Delta m_i \frac{d^2 \mathbf{r}_i}{dt^2} \cdot \delta \mathbf{r}_i \quad (1.2.17)$$

where \mathbf{r}_i is the position vector of the i th mass element. Let us make an initially arbitrary split of this position vector as follows:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i \quad (1.2.18)$$

where \mathbf{R} represents a pivot point, to be chosen later. Substituting this into equation (1.2.17) results in a rather complex expression

$$\delta W_I = - \sum_i \Delta m_i \left(\frac{d^2 \mathbf{R}}{dt^2} \cdot \delta \mathbf{R} + \frac{d^2 \mathbf{R}}{dt^2} \cdot \delta \mathbf{r}' + \frac{d^2 \mathbf{r}'}{dt^2} \cdot \delta \mathbf{R} + \frac{d^2 \mathbf{r}'}{dt^2} \cdot \delta \mathbf{r}' \right). \quad (1.2.19)$$

This can be reduced to

$$\delta W_I = -M \frac{d^2 \mathbf{R}}{dt^2} \cdot \delta \mathbf{R} - M \frac{d^2 \mathbf{R}}{dt^2} \cdot \delta \mathbf{r}'_{CM} - M \frac{d^2 \mathbf{r}'_{CM}}{dt^2} \cdot \delta \mathbf{R} - \sum_i \Delta m_i \frac{d^2 \mathbf{r}'_i}{dt^2} \cdot \delta \mathbf{r}'_i \quad (1.2.20)$$

where the position of the center of mass relative to the pivot point is

$$\mathbf{r}'_{CM} = \frac{1}{M} \sum_i \Delta m_i \mathbf{r}'_i. \quad (1.2.21)$$

We are now faced with choosing a pivot point that simplifies equation (1.2.20). The most obvious choice is to take the pivot point as the center of mass of the object. With this choice, $\mathbf{r}'_{CM} = 0$, since the position of the center of mass relative to the center of mass is the zero vector. In this case, the inertial virtual work is

$$\delta W_I = -M \frac{d^2 \mathbf{R}_{CM}}{dt^2} \cdot \delta \mathbf{R}_{CM} - I_{CM} \frac{d^2 \phi}{dt^2} \delta \phi \quad (1.2.22)$$

where \mathbf{R}_{CM} is the position of the center of mass of the object relative to the original origin. The first term in equation (1.2.22) is the virtual inertial work that would occur if all of the mass were concentrated at the center of mass. The second term is the virtual work for rotations about the center of mass, such as was computed in section 1.2.2. The variable I_{CM} is the moment of inertia about the center of mass and ϕ is the corresponding rotation angle.

Another choice that is useful in certain situations occurs when the pivot point \mathbf{R} is fixed in space, so that $d^2 \mathbf{R}/dt^2 = 0$ and $\delta \mathbf{R} = 0$. If the mass element located at the pivot point is instantaneously stationary, then equation (1.2.20) reduces to

$$\delta W_I = -I \frac{d^2 \phi}{dt^2} \delta \phi \quad (1.2.23)$$

at that instant, as was done in section 1.2.4 for the disk rolling down the ramp. In that case the pivot point is the point of contact of the disk with the ramp at a particular time and the moment of inertia I is that appropriate to rotations about that point. Since \mathbf{R} is fixed, all of the motion of each mass element is contained in changes in \mathbf{r}_i with time.

In section 1.2.4 we implicitly assumed that the gravitational force, which acts separately on each mass element of the disk, can be considered to act solely on the center of mass, i.e., the center of the disk. This seems obvious in the case of a disk, but is less obvious for a more irregularly shaped object. We now prove that the gravitational force effectively acts on the center of mass of an object regardless of its shape.

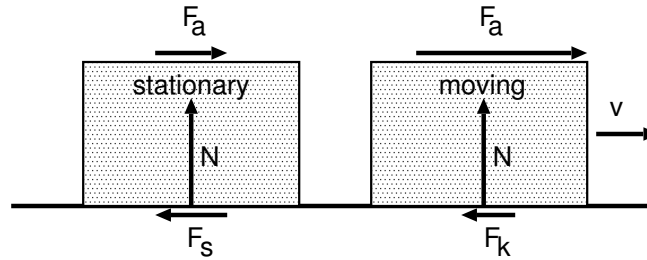


Figure 1.3.1: Two cases of friction. On the left the object is stationary and the frictional force F_s is in balance with the applied force F_a . On the right the object is moving and the frictional force F_k is related to the normal force on the object N . The applied force is not necessarily equal to the frictional force.

The total gravitational work due to small virtual displacements of mass elements can be written

$$\begin{aligned}\delta W_G &= \sum_i dm_i \mathbf{g} \cdot \delta \mathbf{r}_i \\ &= M \mathbf{g} \cdot \delta \mathbf{R}_{CM}\end{aligned}\tag{1.2.24}$$

since the mass-weighted average of the positions of mass elements is simply the center of mass. One note of caution: The above result holds for any force that acts uniformly across the body in question. However, if the force is non-uniform, then the resulting virtual work is more complex. An important example is the total gravitational force on a body in a non-uniform gravitational field.

1.3 Effects of friction

The laws of friction are empirical and are not satisfied exactly. There are two cases, static and kinetic friction. For kinetic friction on an object in contact with a surface, the frictional force is directed opposite the velocity of the object (assuming that the surface is at rest) and has magnitude of

$$F_k = \mu_k N\tag{1.3.1}$$

where N is the normal force exerted by the surface on the object and μ_k is the dimensionless *coefficient of kinetic friction*. In the static case the frictional force is that needed to keep the object stationary in view of the other forces acting on the object. There is an upper limit on the magnitude of the static frictional force equal to

$$F_s \leq \mu_s N\tag{1.3.2}$$

where μ_s is the *coefficient of static friction*. Generally it is true that

$$\mu_s > \mu_k,\tag{1.3.3}$$

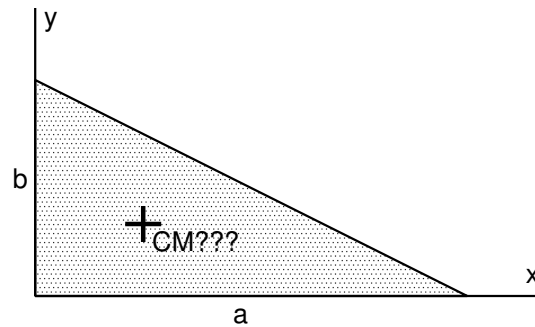


Figure 1.4.1: Triangle for which the center of mass is to be computed.

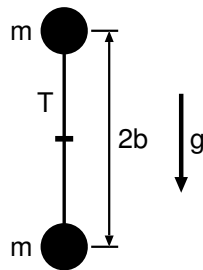


Figure 1.4.2: Two masses attached by a string in a variable gravitational field.

which means that the transition from the static to the kinetic case as the applied force is increased is abrupt, with an instantaneous jump in the acceleration from zero to a finite value. Note that static friction does no virtual work, and thus can be ignored. This is not true with kinetic friction, as displacements of the object occur at the point where the frictional force acts.

1.4 Problems

1. Compute the location of the center of mass of the triangular piece of steel of uniform thickness illustrated in figure 1.4.1.

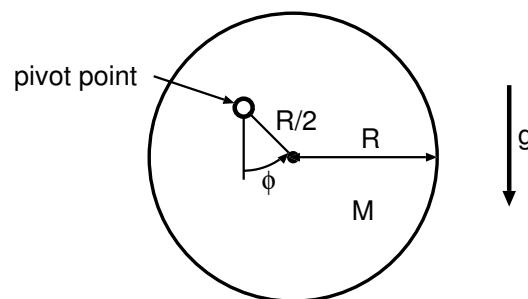


Figure 1.4.3: The disk is free to rotate in the plane of the page around the illustrated pivot point.

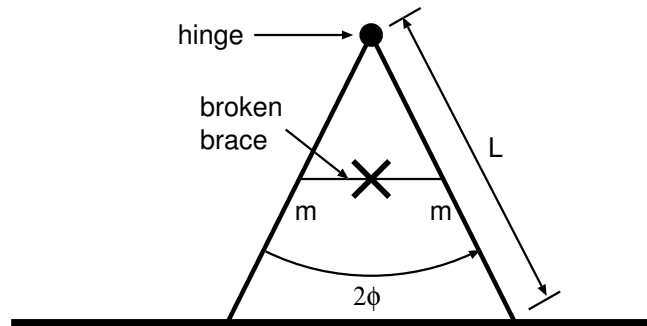


Figure 1.4.4: Step ladder on icy sidewalk with a broken brace

2. Compute the moment of inertia of a solid rod of mass M and length L for:
 - (a) rotation about the center of the rod with the axis of rotation normal to the rod;
 - (b) rotation about one end of the rod with the axis of rotation normal to the rod. Do this both using the above result with the parallel axis theorem and directly.

3. Suppose that the gravitational field decreases with height according to $g = g_0(1 - z/a)$, where g_0 and a are constants. Two equal masses m attached by a thread of length $2b$ are oriented vertically as shown in figure 1.4.2. Answer the following questions using d'Alembert's principle.
 - (a) Compute the vertical acceleration of the system of two masses as a function of height, as measured from the mid-point of the thread.
 - (b) Compute the tension in the thread.

4. A uniform disk of mass M pivots freely about a point half way between the center and edge of the disk. Use d'Alembert's principle to obtain the differential equation governing the rotation angle $\phi(t)$ as illustrated in figure 1.4.3.

5. The cross brace breaks on a step ladder erected on a flat, icy (frictionless) sidewalk as illustrated in figure 1.4.4, causing the ladder to collapse. Fortunately, no one is standing on it at the time. Compute the governing equation for $\phi(t)$ as the ladder collapses using d'Alembert's principle. Assume that both sides of the ladder have mass m and can be idealized as uniform rods of length L . Hints: By symmetry, only one of the rods needs to be considered – its top can be thought to slide frictionlessly down a vertical wall while its bottom slides outward on the floor. Also, the motion of the rod is instantaneously a pure rotation about the intersection of lines originating at the contact points of the rod with the wall and the floor and oriented normal to the instantaneous motions of these points.